Stability of Games with Incomplete Information and a Theorem of Gänssler

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Abstract

Recently, Noguchi (2021) published a paper on the so-called essential stability of the α-core solutions of games with incomplete information in the sense of Milgrom and Weber, based on a stability concept introduced by Fort (1950). In the proof of the main theorem in Noguchi (2021), a part requires a family of probability measures in a compact set to be uniformly and absolutely continuous with respect to some probability measure. To verify this fact, Noguchi (2021) utilized a related result in Gänssler (1971). The present paper aims to examine Gänssler's proof and construct a self-contained simpler proof, taking advantage of dealing only with probability measures instead of a more general class of measures in Gänssler (1971).

Key words: α-core, cooperative games, essential stability, incomplete information, Gänssler

JEL classification: D62, D82

1 Introduction

The α-core is a solution concept for cooperative game theory, initiated by Aumann (1961). It was investigated in a general framework by Kajii (1992) and in the context of incomplete information by Askoura et al. (2013) and Noguchi (2014, 2018).

Recently, Noguchi (2021) published a paper on the so-called essential stability of the α-core solutions of games with incomplete information in the sense of Milgrom and Weber, based on the stability concept introduced by Fort (1950) to investigate the stability of the fixed points of continuous mappings. In the sequel, we refer to games with incomplete information simply as games. There are n players, and a typical player is denoted by $i \in N = \{1, \cdots, n\}$. Each player i is endowed with a set of private information (called a type) T_i and a set of available actions A_i. The payoff to player i, given a type combination $t = (t_1, \cdots, t_n) \in T := T_1 \times \cdots \times T_n$ and an action combination $a = (a_1, \cdots, a_n) \in A := A_1 \times \cdots \times A_n$, is determined by a payoff function $u_i(t, a)$, which incorporates both information and action externalities, that is, the payoff to player i depends on types and actions of all players. We assume player i's type space to be a measurable space $(T_i, \mathcal{T}_i)$ and action space to be a compact metric space A_i. We also assume that the players make decisions in the ex-ante stage, that is, before Nature reveals individual types to the players, and hence none of the players knows his or her type with certainty. This uncertainty is represented by a probability measure (called an information structure) $P \in \mathcal{M}^{+,1}(T, \mathcal{T})$, where $\mathcal{T}$ denotes the product σ-algebra $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n$, where for a measurable space (Ω, Σ), $\mathcal{M}^{+,1}(\Omega, \Sigma)$ denotes the space of probability measures. However, we often omit the σ-algebra Σ for brevity whenever confusion is unlikely.

A strategy available to player i is transition probability $\mu_i : T_i \to \mathcal{M}^{+,1}(A_i, \mathcal{B}(A_i))$, where $\mathcal{B}(A_i)$ denotes the Borel algebra of A_i. A strategy μ_i assigns a mixed action $\mu_i(t_i)$ to each possible type $t_i \in T_i$, and player i enjoys the ex-ante expected payoff $E_i(\mu)$, given a strategy profile $\mu = (\mu_1, \cdots, \mu_n)$, where

$$E_i(\mu) = \int_{T_n \times A_n} \cdots \int_{T_2 \times A_2} \int_{T_1 \times A_1} u_i(t, a)\, d(\mu_1 \otimes P_1)\, d(\mu_2 \otimes P_2) \cdots d(\mu_n \otimes P_n)$$

with P_i being the i-th marginal measure of P. We specify a game by an information structure P and a profile of payoff functions $u = (u_1, \cdots, u_n)$. In this specification of games, the parameter space will be $\mathcal{U} \times \mathcal{M}^{+,1}(T, \mathcal{T})$, where $\mathcal{U}$ is a space of profiles of integrable payoff functions, and $\mathcal{M}^{+,1}(T, \mathcal{T})$ is a space of information structures. Thus in this view, every conceivable game that can occur in the world is represented by a point $(u, P) \in \mathcal{U} \times \mathcal{M}^{+,1}(T, \mathcal{T})$.

An α-core strategy profile is defined as a strategy profile $\mu = (\mu_1, \cdots, \mu_n)$ with which no coalition $S \subset N$ can be unilaterally and strictly better off, given every conceivable retaliatory reaction from the outsiders. The α-core of game (u, P) is defined as the set of all α-core strategy profiles and denoted by $\mathcal{A}(u, P)$. For a technical reason, we identify each strategy profile $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu \otimes P := (\mu_1 \otimes P_1, \mu_2 \otimes P_2, \cdots, \mu_n \otimes P_n) \in \prod_{i \in N} \mathcal{M}^{+,1}(T_i \times A_i)$, where $\mu_i \otimes P_i$ denotes the product measure on $T_i \times A_i$ defined by $\mu_i \otimes P_i (V_i \times W_i) := \int_{V_i} \mu_i(t_i; W_i)\, dP_i(t_i)$, where $V_i \in \mathcal{T}_1$ and $W_i \in \mathcal{B}(A_i)$.

$\mathcal{A}(u, P)$ may be empty for some (u, P). Noguchi (2014, 2018) proved an α-core non-emptiness theorem for finite games with incomplete information. We restrict the class of games (u, P) to be considered as those satisfying the conditions in the above non-emptiness theorem. Each $u_i(t, a)$ is measurable in t, continuous, concave in a, and bounded. In addition, each information structure P has the form $P = k\tau_1 \otimes \cdots \otimes \tau_n$ for some nonnegative measurable function k on T, such that the i-th marginal measure of P equals τ_i and is nonatomic, where $\tau_i \in \mathcal{M}^{+,1}(T_i, \mathcal{T}_i)$. For each $\tau = (\tau_1, \cdots, \tau_n) \in \prod_{i \in N} \mathcal{M}^{+,1}(T_i)$, we use the shorthand notation $\widehat{\tau} = \tau_1 \otimes \cdots \otimes \tau_n \in \mathcal{M}^{+,1}(T, \mathcal{T})$. We specify the information structure by a pair $(k, \widehat{\tau})$, also called an information structure for brevity, and denote the set of information structures in the latter sense by $\mathcal{I}$. Let us keep in mind that we only consider the information structures of the form $P = k\widehat{\tau}$ and that a strategy profile $\mu \otimes P$ admits the expression $\mu \otimes k\widehat{\tau} = \mu \otimes \tau = (\mu_1 \otimes \tau_1, \mu_2 \otimes \tau_2, \cdots, \mu_n \otimes \tau_n)$.

Generally, a solution in the solution set of a game, such as $\mathcal{A}(u, (k, \widehat{\tau}))$, is stable if the solution does not abruptly sift as the game infinitesimally fluctuates around the original one. A game in parameter space, such as $\mathcal{U} \times \mathcal{I}$, is said to be stable if all its solutions are stable, and it is said to be essentially stable if generic games are stable.

Fort (1951) argued that the lower semi-continuity for correspondences provided a suitable notion to describe stability. In light of the following celebrated theorem by Fort (1951), Noguchi (2021) found suitable topologies on relevant probability spaces and pursued a set of conditions for the α-core correspondence $\mathcal{A} : \mathcal{U} \times \mathcal{I} \to \prod_{i \in N} \mathcal{M}^{+,1}(T_i \times A_i)$ to be essentially lower semi-continuous.

In the proof of the main theorem in Noguchi (2021), a part requires a family of probability measures in a compact set to be uniformly and absolutely continuous with respect to some probability measure. To verify this fact, a related result in Gänssler (1971), the proof of which is sketchy, was utilized. The primary purpose of this paper is to revisit and examine Gänssler's proof to construct a self-contained simpler proof, taking advantage of dealing only with probability measures instead of a more general class of measures as in Gänssler (1971).

The present paper is organized as follows. Section 1 introduces Milgrom-and-Weber-type games with incomplete information and a notion of stability applicable to them, posing the problem addressed herein. Section 2 provides an accurate description of the problem in precise mathematical language. Section 3 gives a self-contained proof of a theorem of Gänssler (1971) utilized by Noguchi (2021). Section 4 discusses a further development, which partially contributes to resolving the problem identified by Noguchi (2021).

2 A precise problem statement

In this section, we describe how Noguchi (2021) used Gänssler's (1971) theorem. First, we state Fort's (1951) theorem, a fundamental theorem that allows us to discuss the stability of many game situations of different kinds.

Theorem 1 (Fort, 1951) Let X be a topological space and Y a metric space. Suppose that a correspondence $\mathcal{A} : X \twoheadrightarrow Y$ is nonempty, compact valued, and upper semi-continuous. Then, there exists a countable intersection Q of open dense subsets of X such that $\mathcal{A}$ is lower semi-continuous at every point in Q. If X is a complete metric space, then Q will be a dense residual of X.

In our setup, X corresponds to $\mathcal{U} \times \mathcal{I}$, and Y to *range* $\mathcal{A} \subset \prod_{i \in N} \mathcal{M}^{+,1}(T_i \times A_i)$. We introduce the following two closely related topologies: the s-topology on the set of probability measures on an abstract measurable space and the ws-topology on the set of probability measures on the product of a measurable space and a topological space.

Definition 2 Let (Ω, Σ) be a measurable space. The *strong topology* (s-topology for short) on $\mathcal{M}^{+,1}(\Omega)$ is the coarsest topology for which all functionals $\nu \mapsto \nu(V), V \in \Sigma$, are continuous. The s-topology on $\mathcal{M}^{+,1}(\Omega)$ is the subspace topology on $\mathcal{M}^{+,1}(\Omega) \subset [0,1]^{\Sigma}$, where the latter space is endowed with the topology of pointwise convergence (or the product topology). $\mathcal{M}^{+,1}(\Omega)_s$

indicates that $\mathcal{M}^{+,1}(\Omega)$ is endowed with the s-topology.

Definition 3 Let (Ω, Σ) be a measurable space and X a topological space. The ws-topology on $\mathcal{M}^{+,1}(\Omega \times X)$ is the coarsest topology for which all functionals $\sigma \mapsto \int_{V \times X} c d\sigma$, $V \in \Sigma$, $c \in \mathcal{C}_b(X)$ (the set of bounded continuous functions on X) are continuous. As Balder (2001, p. 496) remarked, the s-topology is the finest topology on $\mathcal{M}^{+,1}(\Omega)$ for which the marginal projection Pr : $\mathcal{M}^{+,1}(\Omega \times X)_{ws} \to \mathcal{M}^{+,1}(\Omega)$ is continuous.

In what follows, we assume that $\mathcal{M}^{+,1}(T_i \times A_i)$ and $\mathcal{M}^{+,1}(T \times A)$ are given the ws-topology and $\mathcal{M}^{+,1}(T_i)$ and $\mathcal{M}^{+,1}(T)$ the s-topology.

We restate the main theorem in Noguchi (2021) below.

Theorem 4 Under Assumptions (A.1)-(A.5), there is a dense residual Q of $\mathcal{U} \times \mathcal{I}$ such that every $(u, (k, \widehat{\tau})) \in Q$ is essential relative to $\mathcal{U} \times \mathcal{I}$.

(A.1) For each $i \in N$, A_i is a compact metric space and a convex subset in a linear space.

(A.2) For each $i \in N$, $\mathcal{P}_i$ consists of nonatomic probability measures.

(A.3) For each $u = (u_1, \cdots, u_n) \in \mathcal{U}$, each u_i is a bounded continuous concave integrand.

(A.4) For each $i \in N$, T_i is a separable measurable space, namely a measurable space with a countably generated σ-algebra.

(A.5) $\widehat{\mathcal{P}} := \{\widehat{\tau} : \forall i \in N \ \tau_i \in \mathcal{P}_i\}$ is compact in $\mathcal{M}^{+,1}(T)_s$.

Some remarks are in order. A dense residual Q is the complement of a topologically negligible set in $\mathcal{U} \times \mathcal{I}$, and $\mathcal{P}_i \subset \mathcal{M}^{+,1}(T_i)$ is an exogenously given set of nonatomic marginal measures τ_i, constituting the information structures $(k, \widehat{\tau})$.

Noguchi (2021) proves the above theorem in the following three steps:

(Step 1) $\mathcal{U} \times \mathcal{I}$ is a complete metric space.

(Step 2) $\prod_{i \in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i)$ admits a metric topology; with the metric topology, $\prod_{i \in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i)$ is compact, where $\mathrm{Pr}_i : \mathcal{M}^{+,1}(T \times A) \to \mathcal{M}^{+,1}(T_i)$ is the marginal projection map onto T_i.

(Step 3) $\mathcal{A} : \mathcal{U} \times \mathcal{I} \to \prod_{i \in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i)$ is upper semi-continuous, where $\prod_{i \in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i)$ is compact from (Step 2). We prove this by showing that $\mathcal{A}$ is closed, that is, the graph of $\mathcal{A}$ is closed.

The present paper concerns the part in (Step 3) that verifies the graph of $\mathcal{A}$ as closed. For each $(u, (k, \widehat{\tau})) \in \mathcal{U} \times \mathcal{I}$ and $S \subset N$, we define $\mathcal{A}_S(u, (k, \widehat{\tau}))$ to be the set of all strategy profiles $\mu \otimes \tau$ that cannot be blocked by S. Then since $\mathcal{A}(u, (k, \widehat{\tau})) = \bigcap_{S \subset N} \mathcal{A}_S(u, (k, \widehat{\tau}))$, it suffices to show that the graph of $\mathcal{A}_S$ is closed.

The defining equation for $\mathcal{A}_S$ involves an extended expected payoff function $\widetilde{E}_i : (\mathcal{U} \times \mathcal{I}) \times \prod_{i \in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i) \to \mathbb{R}$ defined by

$$\widetilde{E}_i(u, (k, \widehat{\tau}), \mu \otimes \nu) = \int_{T \times A} u_i k d(\widehat{\mu} \otimes \widehat{\nu}),$$

where $\widehat{\mu}_t \equiv \mu_{1,t_1} \otimes \cdots \otimes \mu_{n,t_n} : T \to \mathcal{M}^{+,1}(A)$ is the obvious product transition probability. As demonstrated by Noguchi (2021), if $\widetilde{E}_i$ is continuous, then the graph of $\mathcal{A}_S$ is closed. It is clear from the definition of $\widetilde{E}_i$ that the following proposition in Noguchi (2021) plays a crucial role in showing that $\widetilde{E}_i$ is continuous. The proof relies on a variant of the implication (i) $\Rightarrow$ (iii) in Theorem 2.6 in Gänssler (p. 130, 1971), which the present paper focuses on.

Proposition 5 Let $\prod_{i\in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i) \subset \prod_{i\in N} \mathcal{M}^{+,1}(T_i \times A_i)_{ws}$. Then, $\mu \otimes \tau \to \widehat{\mu} \otimes \widehat{\tau}$ defines a continuous mapping from $\prod_{i\in N} \mathrm{Pr}_i^{-1}(\mathcal{P}_i)$ to $\mathcal{M}^{+,1}(T \times A)_{ws}$.

The proof proceeds as follows. Without loss of generality, we assume $n=2$. We first show that if $\mu^\alpha \otimes \tau^\alpha \to \mu^\infty \otimes \tau^\infty$ in $\mathrm{Pr}_1^{-1}(\mathcal{P}_1) \times \mathrm{Pr}_2^{-1}(\mathcal{P}_2) \subset \mathcal{M}^{+,1}(T_1 \times A_1)_{ws} \times \mathcal{M}^{+,1}(T_2 \times A_2)_{ws}$, then $\int_{(V_1\times V_2)\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) \to \int_{(V_1\times V_2)\times A} cd\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)$ for each measurable rectangle $V_1 \times V_2 \in \mathcal{T}_1 \otimes \mathcal{T}_2$ and each c in the set of continuous functions $\mathcal{C}(A_1 \times A_2)$, which follows from an argument similar to that of Theorem 2.5 in Balder (1988, p. 271). We then extend the above convergence on the set of measurable rectangles $V_1 \times V_2 \in \mathcal{T}_1 \otimes \mathcal{T}_2$ to the entire $\mathcal{T}_1 \otimes \mathcal{T}_2$ to complete the proof. For V, $V_1 \times V_2 \in \mathcal{T}_1 \otimes \mathcal{T}_2$, we have

$$\begin{aligned}\left|\int_{V\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) - \int_{V\times A} cd\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)\right| &\leq \|c\|_\infty \left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right)((V \triangle (V_1 \times V_2)) \times A) \\ &+ \left|\int_{(V_1\times V_2)\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) - \int_{(V_1\times V_2)\times A} cd\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)\right| \\ &+ \|c\|_\infty \left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)((V \triangle (V_1 \times V_2)) \times A).\end{aligned}$$

Gänssler's theorem in the present context ensures that there exists a $\lambda \in \mathcal{M}^{+,1}(T_1 \times T_2)$ such that $\forall \epsilon > 0\, \exists \delta > 0\, \forall \widehat{\tau^\alpha}\, \forall V \in \mathcal{T}_1 \otimes \mathcal{T}_2 \left(\lambda(V) < \delta \Longrightarrow \left|\widehat{\tau^\alpha}(V)\right| < \epsilon\right)$. Given $\epsilon > 0$, if $\|c\|_\infty = 0$, choose any $V_1 \times V_2$ and α_0 such that $\alpha \geq \alpha_0 \Longrightarrow \left|\int_{(V_1\times V_2)\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) - \int_{(V_1\times V_2)\times A} cd\sigma^\infty\right| < \epsilon$, and if $\|c\|_\infty \neq 0$, choose $(V_1 \times V_2)$ (see the approximation theorem in Ash, 1972, p. 20) such that $\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)((V \triangle (V_1 \times V_2)) \times A) < \frac{\epsilon}{3}\|c\|_\infty^{-1}$ and for a $\lambda \in \mathcal{M}^{+,1}(T \times A)$ as above, $\lambda((V \triangle (V_1 \times V_2))) < \delta\left(\frac{\epsilon}{3}\|c\|_\infty^{-1}\right)$. We then have

$$\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right)((V \triangle (V_1 \times V_2)) \times A) = \widehat{\tau^\alpha}((V \triangle (V_1 \times V_2))) < \frac{\epsilon}{3}\|c\|_\infty^{-1}$$

for all α. Further, choose α_0 such that

$$\alpha \geq \alpha_0 \Longrightarrow \left|\int_{V'\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) - \int_{V'\times A} cd\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)\right| < \frac{\epsilon}{3}.$$

Then, $\alpha \geq \alpha_0 \Longrightarrow \left|\int_{V\times A} cd\left(\widehat{\mu^\alpha} \otimes \widehat{\tau^\alpha}\right) - \int_{V\times A} cd\left(\widehat{\mu^\infty} \otimes \widehat{\tau^\infty}\right)\right| < \epsilon$ as desired.

3 Gänssler's Theorem

Having understood the importance of the implication (i) ⇒ (iii) in Theorem 2.6 in Gänssler (p. 130, 1971), we examine the original proof, which is sketchy and leaves out details, to see if a self-contained and, more straightforward proof can be obtained, taking advantage of dealing only with probability measures instead of a general class of measures as in Gänssler (1971).

In what follows, for a measurable space (Ω, Σ), $\mathcal{M}(\Omega, \Sigma)$ denotes the set of signed measures and $\mathcal{M}^{+}(\Omega, \Sigma)$ the set of nonnegative measures. We define a notion of uniform, absolute continuity of a family of measures as follows. The s-topology can be defined as well for these spaces as the topology of set-wise convergence.

Definition 6 Let (Ω, Σ) be a measurable space, and let $\Pi \subset \mathcal{M}(\Omega, \Sigma)$. We say $\lambda \in \mathcal{M}^{+}(\Omega, \Sigma)$ *uniformly dominates* Π iff $\forall \epsilon > 0\ \exists \delta > 0\ \forall V \in \Sigma\ \forall \eta \in \Pi \left(\lambda(V) < \delta \Longrightarrow |\eta(V)| < \epsilon\right)$.

Remark 7 The implication (i) ⇒ (iii) in Theorem 2.6 in Gänssler (p. 130, 1971) asserts that if a subset $\Pi \subset \mathcal{M}(\Omega, \Sigma)_s$ is conditionally compact, there exists a $\lambda \in \mathcal{M}^{+}(\Omega, \Sigma)$ that uniformly dominates Π. Since $\mathcal{M}^{+,1}(\Omega, \Sigma)$ is closed in $\mathcal{M}(\Omega, \Sigma)_s$, if Π is conditionally compact in $\mathcal{M}^{+,1}(\Omega, \Sigma)_s$, then the same holds for $\mathcal{M}(\Omega, \Sigma)_s$, and thus there is a $\lambda \in \mathcal{M}^{+}(\Omega, \Sigma)$ that uniformly dominates Π. By normalizing the above λ, we obtained a $\lambda \in \mathcal{M}^{+,1}(\Omega, \Sigma)$ that uniformly dominates Π. Recall that $\widehat{\mathcal{P}}$ is assumed to be compact in $\mathcal{M}^{+,1}(T)_s$ (A.5) and thus is conditionally compact. As mentioned earlier, this is precisely the property needed to prove that $\mu \otimes \tau \to \widehat{\mu} \otimes \widehat{\tau}$ is continuous, where $\widehat{\tau^{\alpha}} \in \widehat{\mathcal{P}}$ for all α.

For a measurable space $(\Omega, \mathcal{G})$, $ba(\Omega, \mathcal{G})$ denotes the set of bounded additive set functions on $\mathcal{G}$, and $\mathcal{M}^{+,1}(\Omega, \mathcal{G})$ the set of probability measures on $\mathcal{G}$.

Lemma 8 Let $(\Omega, \mathcal{G})$ be a measurable space and let $f : \Omega \to \mathbb{R}$ be a bounded measurable function. The f is a uniform limit of simple functions.

Proof. $f(\Omega) \subset \mathbb{R}$ is a separable metric space and is totally bounded. The claim follows from Vakhania et al. (1987, p. 12). ■

Let $B(\Omega, \mathcal{G})$ be the set of uniform limits of simple functions defined for algebras as in Dunford and Schwartz (1958, p. 240). Since our $\mathcal{G}$ is a σ-algebra, $B(\Omega, \mathcal{G})$ is simply the set of bounded measurable functions on Ω.

Lemma 9 $\mathcal{M}^{+,1}(\Omega, \mathcal{G}) \subset ba(\Omega, \mathcal{G}) = B(\Omega, \mathcal{G})^{*}$, where $ba(\Omega, \mathcal{G})$ is endowed with the total variation norm and $B(\Omega, \mathcal{G})^{*}$ denotes the Banach space dual of $B(\Omega, \mathcal{G})$.

Proof. Theorem 1 in Dunford and Schwartz (1958, p. 258). ■

Definition 10 For an algebra $\mathcal{A}$ on Ω, $\mathcal{T}_{\mathcal{A}}$ denotes the topology of pointwise convergence (the product topology) on $[0,1]^{\mathcal{A}}$. For a measurable space $(\Omega,\mathcal{G})$, since $\mathcal{M}^{+,1}(\Omega,\mathcal{G}) \subset [0,1]^{\mathcal{G}}$, $\mathcal{T}_{\mathcal{G}}$ induces a subspace topology, called the s-topology on $\mathcal{M}^{+,1}(\Omega,\mathcal{G})$. We denote $\mathcal{M}^{+,1}(\Omega,\mathcal{G})$ with the s-topology by $\mathcal{M}^{+,1}(\Omega,\mathcal{G})_s$.

Lemma 11 $\mathcal{M}^{+,1}(\Omega,\mathcal{G})_s = \mathcal{M}^{+,1}(\Omega,\mathcal{G})$, *where weak* elenotes the weals star topology.*

Proof. Let $f \in B(\Omega,\mathcal{G})$ and let $\tau^{\alpha} \to \tau^{\infty}$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{G})_s$. Approximate f by a simple f'. Then

$$|\tau^{\alpha}(f) - \tau^{\infty}(f)| \leq |\tau^{\alpha}(f) - \tau^{\alpha}(f')| + |\tau^{\alpha}(f') - \tau^{\infty}(f')| + |\tau^{\infty}(f) - \tau^{\infty}(f')|$$
$$\leq 2\|f - f'\|_{\infty} + |\tau^{\alpha}(f') - \tau^{\infty}(f')|,$$

which implies $0 \leq \liminf_{\alpha} |\tau^{\alpha}(f) - \tau^{\infty}(f)| \leq \limsup_{\alpha} |\tau^{\alpha}(f) - \tau^{\infty}(f)| \leq 2\|f - f'\|_{\infty}$ and hence $\lim_{\alpha} |\tau^{\alpha}(f) - \tau^{\infty}(f)| = 0$. Consequently, the claim holds in light of Lemma 9. ■

Lemma 12 (Gänssler, 1971, Lemma 1.15, p. 127) Let $X_0 \subset X \subset Y$ be topological spaces, where Y is compact Hausdorff. Then X_0 is conditionally compact in X iff $cl_Y X_0 \subset X$.

Proof. Suppose X_0 is conditionally compact in X. Then $\exists$ a compact $C \subset X$ s.t. $X_0 \subset C \subset X$. Then C is compact in Y and hence closed in Y. Thus $cl_Y X_0 \subset C \subset X$. Conversely, if $cl_Y X_0 \subset X$, $cl_Y X_0$ is compact in X since $cl_Y X_0$ is compact in Y, and hence, X_0 is conditionally compact in X. ■

Lemma 13 (Gänssler, 1971, Lemma 1.19, p. 127) Let $\mathcal{O}_1 \subset \mathcal{O}_2$ be topologies on X, where $\mathcal{O}_1$ is Hausdorff. Let $X_0 \subset X$. For a topology $\mathcal{O}$ on X, $cl_{\mathcal{O}}X_0$ denotes the closure of X_0 with respect to $\mathcal{O}$. If a subset X_0 of X is conditionally compact in $(X,\mathcal{O}_2)$, then X_0 is also conditionally compact in $(X,\mathcal{O}_1)$ and $cl_{\mathcal{O}_1}X_0 = cl_{\mathcal{O}_2}X_0$. Furthermore, $\mathcal{O}_1 = \mathcal{O}_2$ on $cl_{\mathcal{O}_1}X_0$.

Proof. Suppose X_0 is conditionally compact in $(X,\mathcal{O}_2)$. Then since the identity map $(X,\mathcal{O}_2) \to (X,\mathcal{O}_1)$ is continuous, X_0 is conditionally compact in $(X,\mathcal{O}_1)$. Further, since $cl_{\mathcal{O}_1}X_0$ is closed in $(X,\mathcal{O}_2)$, we have $cl_{\mathcal{O}_2}X_0 \subset cl_{\mathcal{O}_1}X_0$, and since $cl_{\mathcal{O}_2}X_0$ is compact in $(X,\mathcal{O}_2)$, it is compact in $(X,\mathcal{O}_1)$ and hence closed. Thus, the other inclusion $cl_{\mathcal{O}_1}X_0 \subset cl_{\mathcal{O}_2}X_0$ holds as well. Note that $cl_{\mathcal{O}_2}X_0 \to cl_{\mathcal{O}_1}X_0$ is a homeomorphism. ■

For a family of subsets $\mathcal{S}$ of Ω, $\sigma[\mathcal{S}]$ denotes the smallest σ-algebra containing $\mathcal{S}$. Let $(\Omega,\mathcal{F})$ be a measurable space and $\mathcal{F}_0$ a sub-σ-algebra of $\mathcal{F}$.

Lemma 14 Let R be an algebra s.t. $\sigma[R] = \mathcal{F}_0$. Let $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0) \xrightarrow{j} [0,1]^{R}$ be the restriction map defined by $j:\tau\mapsto\tau|_R$. Then j is an injection, and

$$\mathcal{O}_R=\left\{j^{-1}(U):U\text{ is open in }[0,1]^R\text{ with the product topology}\right\}$$

defines a Hausdorff topology on $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$. $\mathcal{O}_R$ is the coarsest topology on $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$ with the property that j is continuous and hence that $\tau_\alpha\to\tau$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$ with $\mathcal{O}_R$ iff $\tau_\alpha|_R\to\tau|_R$. We also have $\mathcal{O}_R\subset$ the s-topology.

Proof. Since $[0,1]^R$ is a Hausdorff space, the unique extension theorem (Dunford and Schwartz, 1958, Corollary 9, p. 136) implies that j is a continuous injection. ■

Let Π be a subset of $\mathcal{M}^{+,1}(\Omega,\mathcal{F})$. When $\mathcal{A}$ is a sub-algebra of $\mathcal{F}$, $\Pi|_{\mathcal{A}}$ denotes the obvious restriction.

Lemma 15 Suppose $\Pi|_{\mathcal{F}_0}$ is conditionally compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$, and let $\{\tau_n\}\subset\Pi|_{\mathcal{F}_0}$. Then $\tau_0\in[0,1]^{\mathcal{F}_0}$ defined by $\tau_0(E)=\limsup_n\tau_n(E)$ for all $E\in\mathcal{F}_0$ is a member of $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$ and $\tau_n\to\tau_0$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$.

Proof. Since $\mathcal{O}_R$ is Hausdorff (Lemma 14), $\mathcal{O}_R\subset$ the s-topology on $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$, and $\Pi|_{\mathcal{F}_0}$ is conditionally compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$, Lemma 13 implies that $cl\,\Pi|_{\mathcal{F}_0}=cl_{\mathcal{O}_R}\,\Pi|_{\mathcal{F}_0}$ and that $\mathcal{O}_R$ is the s-topology on $cl_{\mathcal{O}_R}\,\Pi|_{\mathcal{F}_0}$. Define $\tau_0\in[0,1]^{\mathcal{F}_0}$ by $\tau_0(E)=\limsup_n\tau_n(E)$ for all $E\in\mathcal{F}_0$. Then since $\Pi|_{\mathcal{F}_0}$ is conditionally compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s\subset[0,1]^{\mathcal{F}_0}$, Lemma 12 implies that $\tau_0\in cl_{[0,1]^{\mathcal{F}_0}}\,\Pi|_{\mathcal{F}_0}\subset\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$, where $cl_{[0,1]^{\mathcal{F}_0}}\,\Pi|_{\mathcal{F}_0}=cl\,\Pi|_{\mathcal{F}_0}=cl_{\mathcal{O}_R}\,\Pi|_{\mathcal{F}_0}$. Since $\lim_n\tau_n(E)=\tau_0(E)$ for all $E\in R$, we have $\tau_n|_R\to\tau_0|_R$, i.e., $\tau_n\to\tau_0$ in $cl_{\mathcal{O}_R}\,\Pi|_{\mathcal{F}_0}$ with $\mathcal{O}_R$, and hence $\tau_n\to\tau_0$ in $cl_{\mathcal{O}_R}\,\Pi|_{\mathcal{F}_0}$ with the s-topology. Thus $\tau_n\to\tau_0$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s$ as desired. ■

Remark 16 Lemma 15 is comparable to Corollary 2.4 in Gänssler (1971, p. 129). Since we work with the set of countably additive set functions, $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$, whereas Gänssler works with $ca(\Omega,\mathcal{F}_0)$, we can take advantage of the inclusion $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)_s\subset[0,1]^{\mathcal{F}_0}$, which significantly simplifies the proof, leading to a stronger conclusion that τ_0 is a member of $\mathcal{M}^{+,1}(\Omega,\mathcal{F}_0)$.

Proposition 17 If Π is conditionally compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$ then Π is conditionally sequentially compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$, that is, every sequence in Π has a convergent subsequence with a limit in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})$.

Proof. Let $\{\tau_n\}\subset\Pi$ be a sequence. Define $\lambda\in\mathcal{M}^{+,1}(\Omega,\mathcal{F})$ by $\lambda=\sum_n 2^{-n}\tau_n$. Then each τ_n is λ-continuous. Let $p_n\in L_1(\Omega,\mathcal{F},\lambda)$ be the Radon-Nikodým derivative of τ_n, where we denote a representative of p_n by the same symbol. Consider $\Psi=(p_1,\cdots,p_n):\Omega\to\mathbb{R}^n$. Then $p_i=\pi_i\circ\Psi$, where $\pi_i:\mathbb{R}^n\to\mathbb{R}$ is the i-th projection. Note that $\Psi^{-1}(\mathcal{B}(\mathbb{R}^n))\subset\mathcal{F}$ is countably generated since $\mathcal{B}(\mathbb{R}^n)$ is. Further, p_i is $\Psi^{-1}(\mathcal{B}(\mathbb{R}^n))$-measurable for each $i \in N$. Denote $\mathcal{F}_0 := \Psi^{-1}(\mathcal{B}(\mathbb{R}^n)) \subset \mathcal{F}$. Since $\mathcal{M}^{+,1}(\Omega, \mathcal{F})_s \to \mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_s$, defined by $\tau \mapsto \tau|_{\mathcal{F}_0}$, is continuous, $\Pi|_{\mathcal{F}_0}$ is conditionally compact in $\mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_s$. Let R be a countable generator of $\mathcal{F}_0$, which may be assumed to be a sub-algebra of $\mathcal{F}_0$. Then, by the diagonal process, we find a subsequence, denoted by $\{\tau_n\}$ as before, which is convergent on R, that is, satisfies $\tau_n|_{\mathcal{F}_0}(V') \to \tau_0(V')$ for each $V' \in R$, where $\tau_0 \in [0,1]^{\mathcal{F}_0}$ is defined as in Lemma 15. Then by Lemma 15, $\tau_n|_{\mathcal{F}_0} \to \tau_0$ in $\mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_s$. Let $V \in \mathcal{F}$ and $V_0 \in \mathcal{F}_0$. Then

$$\begin{aligned}\tau_n(V \cap V_0) &= \int p_n 1_{V \cap V_0} d\lambda \\ &= \int E(p_n 1_{V \cap V_0} \mid \mathcal{F}_0) d\lambda \\ &= \int E(1_V \mid \mathcal{F}_0) 1_{V_0} p_n d\lambda \\ &= \int E(1_V \mid \mathcal{F}_0) 1_{V_0} d\tau_n,\end{aligned}$$

where $E(\cdot \mid \mathcal{F}_0) : L_1(\Omega, \mathcal{F}, \lambda) \to L_1(\Omega, \mathcal{F}_0, \lambda_{\mathcal{F}_0})$ is the conditional expectation operator. Letting $V_0 = \Omega$ we obtain

$$\tau_n(V) = \int E(1_V \mid \mathcal{F}_0) d\tau_n = \int E(1_V \mid \mathcal{F}_0) d\, \tau_n|_{\mathcal{F}_0}$$

for each $V \in \mathcal{F}$. Observe that $E(1_V \mid \mathcal{F}_0) \in B(\Omega, \mathcal{F}_0)$, which can be seen as follows. Since 1_Ω is $\mathcal{F}_0$-measurable and $1_V \leq 1_\Omega$, we have $0 \leq E(1_V \mid \mathcal{F}_0) \leq E(1_\Omega \mid \mathcal{F}_0) = 1_\Omega \ \lambda - a.e.$ By Lemma 11 we have $\mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_s = \mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_{weak^*}$, and since $\tau_n|_{\mathcal{F}_0} \to \tau_0$ in $\mathcal{M}^{+,1}(\Omega, \mathcal{F}_0)_s$ as shown above, $\tau_n(V)$ is convergent for each $V \in \mathcal{F}$. Then by Corollary 4 in Dunford and Schwartz (1958, p. 160), $\tau_\infty(V) := \lim_n \tau_n(V)$, $V \in \mathcal{F}$, defines a measure on $\mathcal{F}$, which is a member of $\mathcal{M}^{+,1}(\Omega, \mathcal{F})$. ■

Remark 18 If also Π is s-closed, $\tau_\infty \in \Pi$, and hence Π is sequentially compact.

Remark 19 If Π is Suslin compact, it is metrizable and hence is sequentially compact. See Schwartz (1973, Corollary 2, p. 106).

The idea of the following proof is taken partly from the proof of Theorem 2 in Dunford and Schwartz (1958, p. 306) and Theorem 2.6 in Gänssler (1971, p. 130). We reiterate them in a simple, streamlined manner to be as self-contained as possible.

Theorem 20 If Π is conditionally compact in $\mathcal{M}^{+,1}(\Omega, \mathcal{F})_s$ then Π is uniformly dominated by some $\lambda \in \mathcal{M}^{+,1}(\Omega, \mathcal{F})$, that is, $\forall \epsilon > 0\ \exists \delta > 0\ \forall \tau \in \Pi\ \forall V \in \mathcal{F}\ (\lambda(V) < \delta \Longrightarrow \tau(V) < \epsilon)$.

Proof. We will find a $\lambda \in \mathcal{M}^{+,1}(\Omega, \mathcal{F})$ that dominates Π, that is, $\forall \epsilon > 0\ \forall \tau \in \Pi\ \exists \delta(\tau) > 0.$ $\forall V \in \mathcal{F}\ (\lambda(V) < \delta(\tau) \Longrightarrow \tau(V) < \epsilon)$ and argue that it, in fact, uniformly dominates Π. We first claim that $\forall\epsilon>0\,\exists\{\tau_1,\cdots,\tau_n\}\subset\Pi\,\exists\delta>0\,\forall\tau\in\Pi\,\forall V\in\mathcal{F}\,(\forall i\ \tau_i(V)<\delta\Longrightarrow\tau(V)<\epsilon)$. Suppose the claim is false. Then $\exists\epsilon>0\,\forall\tau_1\in\Pi\,\exists\tau_2\in\Pi$ and $\exists V_1\in\mathcal{F}$ s.t.

$$\tau_1(V_1)<\frac{1}{2},\tau_2(V_1)\geq\epsilon,$$

$\exists\tau_3\in\Pi$ and $\exists V_2\in\mathcal{F}$ s.t.

$$\tau_1(V_2)<\frac{1}{2^2},\tau_2(V_2)<\frac{1}{2^2},\tau_3(V_2)\geq\epsilon,$$

and by continuing in this manner, we find a sequence $\{\tau_n\}\subset\Pi,\{V_n\}\subset\mathcal{F}$ s.t.

$$\tau_1(V_n)<\frac{1}{2^n},\cdots,\tau_n(V_n)<\frac{1}{2^n},\tau_{n+1}(V_n)\geq\epsilon.$$

Since Π is conditionally sequentially compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$ (Proposition 17), there exists a subsequence of $\{\tau_n\}$, which we will denote by $\{\tau_n\}$ as before, such that $\tau_n\to\tau$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$. Define $\lambda_0\in\mathcal{M}^{+,1}(\Omega,\mathcal{F})$ by $\lambda_0=\sum_n 2^{-n}\tau_n$. Then each τ_n is λ_0-continuous. Since $\lim_n\tau_n(V)$ exists for each $V\in\mathcal{F}$, the Vitali-Hahn-Saks theorem (Dunford and Schwartz, 1958, p. 158) implies that $\lim_{\lambda_0(V)\to0}\tau_m(V)=0$ uniformly in m, that is, $\forall\epsilon>0\,\exists\delta>0\,\left(\lambda_0(V)<\delta\Longrightarrow\sup_m\tau_m(V)<\epsilon\right)$. Then since

$$\lambda_0(V_n)\leq\sum_{i=1}^{n}\frac{1}{2^i}\frac{1}{2^n}+\sum_{i=n+1}^{\infty}\frac{1}{2^i}<\frac{1}{2^{n-1}},$$

choose N s.t. $n\geq N\Longrightarrow\frac{1}{2^{n-1}}<\delta$. Then $\forall n\geq N\ \sup_m\tau_m(V_n)<\epsilon$, that is, $\lim_n\sup_m\tau_m(V_n)=0$, which contradicts $\tau_{n+1}(V_n)\geq\epsilon>0$ since $\tau_{n+1}(V_n)\leq\sup_m\tau_m(V_n)\to0$. Thus for each n, we can find $\left\{\tau_1^{(n)},\cdots,\tau_{m_n}^{(n)}\right\}\subset\Pi$ and $\exists\delta_n>0\ \forall\tau\in\Pi\ \forall V\in\mathcal{F}\ \left(\forall i\ \tau_i^{(n)}(V)<\delta_n\Longrightarrow\ \tau(V)<\frac{1}{n}\right)$. Since $\left[\sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\frac{1}{2^n}\frac{1}{2^i}\tau_i^{(n)}\right](\Omega)=\sum_{n=1}^{\infty}\frac{1}{2^n}\left(1-\frac{1}{2^{m_n}}\right)$, where $1\leq m_n$, we obtain $\frac{1}{2}\leq\left[\sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\frac{1}{2^n}\frac{1}{2^i}\tau_i^{(n)}\right](\Omega)\leq1$. Define $\lambda\in\mathcal{M}^{+,1}(\Omega,\mathcal{F})$ by normalizing $\sum_{n=1}^{\infty}\sum_{i=1}^{m_n}\frac{1}{2^n}\frac{1}{2^i}\tau_i^{(n)}$. If $\lambda(V)=0$ then for each n, we have $\tau_i^{(n)}(V)=0$ for each $i=1,\cdots,m_n$ and hence $\forall\tau\in\Pi\ \tau(V)<\frac{1}{n}$. Thus if $\lambda(V)=0$ then $\forall\tau\in\Pi\ \tau(V)=0$, which confirms that λ dominates Π. Now, suppose λ does not uniformly dominate Π. Then $\exists\epsilon>0\,\forall n\,\exists V_n\in\mathcal{F}\,\exists\tau_n\in\Pi$ s.t. $\lambda(V_n)<\frac{1}{n}$ and $\tau_n(V_n)\geq\epsilon$. Since Π is conditionally sequentially compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$, $\{\tau_n\}$ has a convergent subsequence $\tau_{n_i}\to\tau_0$ in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$. Then, using the Vitali-Hahn-Saks theorem (Dunford and Schwartz, 1958, p. 158), we deduce that $\lim_{\lambda(V)\to0}\sup_i\tau_{n_i}(V)=0$, that is, $\forall\epsilon>0\,\exists\delta>0\left(\lambda(V)<\delta\Longrightarrow\sup_i\tau_{n_i}(V)<\epsilon\right)$. Choose I s.t. $i\geq I\Longrightarrow\frac{1}{n_i}<\delta$. Then $\forall i\geq I\ \sup_i\tau_{n_i}(V_{n_i})<\epsilon$, which is impossible since $\epsilon\leq\tau_{n_i}(V_{n_i})\leq\sup_i\tau_{n_i}(V_{n_i})\to0$. ■

Remark 21 The last part of the proof shows that if Π is conditionally sequentially compact in $\mathcal{M}^{+,1}(\Omega,\mathcal{F})_s$ then

$$\exists\lambda \in \mathcal{M}^{+,1}(\Omega,\mathcal{F})\,\forall\tau \in \Pi\ \forall V \in \mathcal{F}\ \lim_{\lambda(V)\to 0}\tau(V) = 0$$

$$\Longrightarrow \exists\lambda \in \mathcal{M}^{+,1}(\Omega,\mathcal{F})\ \forall V \in \mathcal{F}\ \lim_{\lambda(V)\to 0}\sup_{\tau\in\Pi}\tau(V) = 0.$$

Remark 22 Let $ca(\Omega,\mathcal{G}) \subset ba(\Omega,\mathcal{G})$ be the set of countably additive R-valued set functions on $\mathcal{G}$, where $\mathcal{G}$ is a σ-algebra on Ω. Let $\lambda \in ca(\Omega,\mathcal{G})$ be nonnegative and $\Pi \subset ca(\Omega,\mathcal{G})$. According to Gänssler (1971), Π is said to be dominated by λ if

(i) $\forall\mu \in \Pi\ \forall E \in \mathcal{G}\ \big(\lambda(E) = 0 \Longrightarrow \mu(E) = 0\big)$, which is equivalent to

(ii) $\forall\mu \in \Pi\ \forall\epsilon > 0\ \exists\delta(\mu,\epsilon) > 0\ \forall E \in \mathcal{G}\ \big(\lambda(E) < \delta(\mu,\epsilon) \Longrightarrow \mu(E) < \epsilon\big)$.

The above equivalence follows from Lemma 13 in Dunford and Schwartz (1958, p. 131). These statements have the following equivalent expressions:

(iii) $\forall\mu \in \Pi\ \lim_{\lambda(E)\to 0}\mu(E) = 0$.

(iv) $\forall\mu \in \Pi$, μ is λ-continuous.

Moreover, the following statements are equivalent:

(i) $\forall\epsilon > 0\ \exists\delta(\epsilon) > 0\ \forall\mu \in \Pi\ \forall E \in \mathcal{G}\ \big(\lambda(E) < \delta(\epsilon) \Longrightarrow \mu(E) <\big)\epsilon$.

(ii) $\lim_{\lambda(E)\to 0}\sup_{\mu\in\Pi}\mu(E) = 0$.

4 Further development

Regarding condition (A.5), which states that $\widehat{\mathcal{P}} := \{\widehat{\tau} : \forall i \in N\ \tau_i \in \mathcal{P}_i\}$ is compact in $\mathcal{M}^{+,1}(T)_s$, Noguchi (2021) remarked that it is more stringent than demanding each $\mathcal{P}_i$ to be compact in $\mathcal{M}^{+,1}(T_i)_s$, while the compactness of $\widehat{\mathcal{P}}$ implies that of $\mathcal{P}_i$ since the marginal projection $\widehat{\mathcal{P}} \to \mathcal{P}_i$ is continuous and surjective. Proving the converse to the above statement was posed in Noguchi (2021) to be addressed in future research. In this section, we will demonstrate that with (A.4), which requires each σ-algebra $\mathcal{T}_i$ to be countably generated, if each $\mathcal{P}_i$ is compact in $\mathcal{M}^{+,1}(T_i)_s$ then $\mathcal{P}_i$ is metrizable, which will bring us one step closer to solving the above problem.

Theorem 23 (Kuratowski, 1968, p. 22) Let X be a compact metric space and Y be a Hausdorff topological space such that $Y = f(X)$ for some continuous map $f : X \to Y$. Then Y is metrizable.

Proof. Since Y is regular (even normal), it suffices to show that the topology $\mathcal{O}_Y$ of Y has a countable basis. Let $B_1, B_2 \cdots$ be a countable basis for the topology of X, and let $W_1, W_2, \cdots$ be finite unions of $B_1, B_2 \cdots$. Let $y \in V \in \mathcal{O}_Y$. We wish to find W_n such that $y \in f(W_n^c)^c \subset V$ to show that $f(W_1^c)^c, f(W_2^c)^c, \cdots$ form a basis for $\mathcal{T}_Y$. To this end, we find W_n such that $f^{-1}(y) \subset W_n$ and $f^{-1}(V^c) \cap W_n = \varnothing$. Then $f^{-1}(y) \subset W_n \Longleftrightarrow W_n^c \subset f^{-1}(\{y\}^c) \Longleftrightarrow f(W_n^c) \subset \{y\}^c \Longleftrightarrow y \in f(W_n^c)^c$ and $f^{-1}(V^c) \cap W_n = \varnothing \Longleftrightarrow f^{-1}(V) \cup W_n^c = X \Longleftrightarrow V \cup f(W_n^c) = Y \Longleftrightarrow f(W_n^c)^c \subset V$. ■

Proposition 24 Let (Ω, Σ) be a measurable space with a countably generated σ-algebra Σ, and let $\Pi \subset \mathcal{M}^{+,1}(\Omega, \Sigma)_s$ be a conditionally compact subset. Then Π is metrizable.

Proof. Let $K \subset \mathcal{M}^{+,1}(\Omega, \Sigma)_s$ be a compact subset such that $\Pi \subset K$. Let A be a compact metric space, and $\Pr : \mathcal{M}^{+,1}(\Omega \times A, \Sigma \otimes \mathcal{B}(A))_{ws} \rightarrow \mathcal{M}^{+,1}(\Omega, \Sigma)_s$ be the marginal projection map onto Ω. Proposition 2.3 in Balder (2001, p. 500) ensures that $\Pr^{-1}(K) \subset \mathcal{M}^{+,1}(\Omega \times A, \Sigma \otimes \mathcal{B}(A))_{ws}$ is metrizable, and Theorem 2.5 (p. 505) implies that $\Pr^{-1}(K) \subset \mathcal{M}^{+,1}(\Omega \times A, \Sigma \otimes \mathcal{B}(A))_{ws}$ is compact as well. Theorem 23 yields our conclusion. ■

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