Theorems of the Alternative and Linear Programming

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1. Introduction

Theorems of the alternative in linear inequalities, which are very useful in mathematics and economics, are proved in various ways. The purpose of the present paper is to prove theorems of the alternative in a unified way by making use of linear programming.

2. Definitions

Let $A$ and $b$ be any given $m \times n$ matrix and $m$-dimensional column vector, respectively, and let $e$ be the $m$-dimensional column vector whose components are all ones, i.e., $e=(1, \ldots, 1)\in R^m$. A prime denotes the transposition of a matrix or a vector. We define for any two vectors $u=(u_i)$ and $v=(v_i)$:

- $u\geq v$ if $u_i\geq v_i$ for all $i$,
- $u\geq v$ if $u\equiv v$ and $u\neq v$,
- $u>v$ if $u_i>v_i$ for all $i$.

A vector $x$ is said to be nonnegative if $x\geq 0$, semipositive if $x\geq 0$, and positive if $x>0$. Moreover, we denote by (I) or (II) the negation of the statement (I) or (II), respectively.

3. Duality Theorems in Linear Programming

For any primal and dual linear programming problems the following lemmas hold:

Lemma 1.² If the primal or the dual problem has an optimal solution, then the other problem also has an optimal solution and the values of the objective functions of both problems are equal.

Lemma 2.³ If a feasible solution exists for the minimum (maximum) problem and the value of

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¹ For applications of theorems of the alternative, see Gale [2, pp. 208, 264, 313], Gale and Nikaido [3], and Uekawa [8], for example.

² See, for example, Dantzig [1, pp. 134–135] and Hadley [5, pp. 229–230].

³ See, for example, Dantzig [1, p. 134] and Goldman and Tucker [4, p. 60].
its objective function has a finite lower (upper) bound, then there exists an optimal solution for the minimum (maximum) problem.

4. Theorems of the Alternative

We prove the following theorems, using the two lemmas above.

Theorem 1 (Gordan’s Theorem).\(^4\) Either

(I) \(Ax>0\) has a solution \(x\),
or

(II) \(A'y=0, \ y\geq 0\) has a solution \(y\).

but never both.

Proof. First, we formulate the following primal and dual linear programming problems:

Minimize \(e'z\)

(P) subject to \(Ax+z\geq e,\)

\[z\geq 0.\]

Maximize \(e'y\)

(D) subject to \(A'y=0,\)

\[y\leq e,\]

\[y\geq 0.\]

(\(\bar{\text{I}}\))\(\Rightarrow\) (\(\text{II}\)). If (I) does not hold, then at least one component of \(Ax\) is nonpositive for any \(x\). In this case to satisfy the constraint of (P) we must have \(z\geq 0\) for any \(x\). From Lemma 2 (P) has an optimal solution \(z^*\) and so \(e'z^*>0\). From Lemma 1 (D) also has an optimal solution \(y^*\) such that \(e'y^*>0, A'y^*=0, y^*\geq 0\). This implies that \(y^*\geq 0\), and hence (II) holds for \(y^*\).

(I)\(\Rightarrow\) (\(\bar{\text{II}}\)). If (I) holds for \(\hat{x}\), i.e., \(Ax^*\geq e\) holds for \(x^*=\alpha x\), where \(\alpha\) is a sufficiently large number. Hence, (P) has an optimal solution \(z=0\) and \(e'z=0\). Thus, from Lemma 1 (D) also has an optimal solution \(y^*\) such that \(e'y^*=0, A'y^*=0, e\geq y^*\geq 0\). Since \(e'y\leq e'y^*=0\) for any feasible \(y\) such that \(A'y=0, e\geq y\geq 0\), we cannot have \(A'y=0, y\geq 0\). Therefore, (\(\bar{\text{II}}\)) is derived. Q. E. D.

Theorem 2 (Farkas’ Lemma).\(^5\) Either

(I) \(Ax=b, \ x\geq 0\) has a solution \(x\),
or

(II) \(A'y\leq 0, b'y>0\) has a solution \(y\),

\(^4\) See, for example, Dantzig[1, pp. 136–137], Gale [2, p. 48], Kemp and Kimura [6, pp. 5–6], and Mangasarian [7, p. 31].

\(^5\) See, for example, Dantzig [1, p. 137], Gale [2, pp. 44–46], Kemp and Kimura [6, p. 3], and Mangasarian [7, pp. 31–32].
**but never both.**

**Proof.** Consider the following primal and dual linear programming problems:

Minimize \( e'(u+v) \)

(P) subject to

\[ \begin{align*}
Ax + u - v &\leq b, \\
x &\geq 0, \ u, v \geq 0.
\end{align*} \]

Maximize \( b'y \)

(D) subject to

\[ \begin{align*}
Ay &\leq 0, \\
y &\leq e, \\
-y &\leq e.
\end{align*} \]

\( \text{(I)} \Rightarrow (\text{II}). \) If (I) does not hold, then the value of the objective function of (P) must be positive for any \( x \geq 0. \) As \( u, v \geq 0, \) it follows from Lemma 2 that (P) has an optimal solution \( u^*, v^* \geq 0 \) and \( e'(u^* + v^*) > 0. \) From Lemma 1 (D) also has an optimal solution \( y^* \) such that \( Ay^* \leq 0, \ b'y^* > 0. \) Therefore, (II) holds for \( y^*. \)

\( (\text{I}) \Rightarrow (\text{II}). \) If (I) holds for \( x \geq 0, \) then in (P) min \( e'(u+v)=0 \) for \( u=v=0. \) So, from Lemma 1 we have \( b'y \leq \max b'y = 0 \) for any \( y \) such that \( Ay \leq 0. \) Thus, (II) is derived. Q. E. D.

**Theorem 3 (Stiemke’s Theorem).** 6

Either

(I) \( Ax \leq 0 \) has a solution \( x, \)

or

(II) \( Ay=0, \ y > 0 \) has a solution \( y, \)

but never both.

**Proof.** Consider the following primal and dual linear programming problems:

Minimize \( e'w \)

(P) subject to

\[ \begin{align*}
Ay &\leq 0, \\
y + w &\leq e, \\
y, w &\geq 0.
\end{align*} \]

Maximize \( e'z \)

(D) subject to

\[ \begin{align*}
Ax + z &\leq 0, \\
z &\leq e, \\
z &\geq 0.
\end{align*} \]

\( (\text{II}) \Rightarrow (\text{I}). \) If (II) does not hold, then \( y > 0 \) for any \( y \) such that \( Ay = 0, \) and thus always \( w \geq 0 \) in (P). It therefore follows from Lemma 2 that there exists an optimal solution \( w^* \geq 0 \) and \( e'w^* > 0 \) in (P). From Lemma 1 there exists an optimal solution in (D) such that \( e'z^* > 0, \ Ax^* + z^* \leq 0, z^* \geq 0. \) This implies that \( z^* \geq 0 \) and hence \( Ax^* \leq 0. \) Accordingly, (I) is obtained.

\( (\text{II}) \Rightarrow (\text{I}). \) If (II) holds for \( y > 0, \) then there is \( y^* \) such that \( Ay^* = 0, y^* \geq e, \) where \( y^* = \alpha \hat{y} \) for a

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6 See, for example, Dantzig [1, p. 138], Gale [2, p. 49], Kemp and Kimura [6, p. 3], and Mangasarian [7, p. 32].
sufficiently large number $\alpha$. Therefore, $w=0$ is an optimal solution of (P). By Lemma 1 there exists $z^* \geq 0$ in (D) such that $e'z^* = 0$. On the other hand, if (I) is true, then always $z \geq 0$ in (D), so that $0 < e'z \leq e'z^* = 0$, a contradiction. Therefore, (I) is derived. Q. E. D.

**Theorem 4 (Ville’s Theorem).** Either

(I) $Ax > 0$, $x \geq 0$ has a solution $x$,

or

(II) $A'y \leq 0$, $y \geq 0$ has a solution $y$,

but never both.

**Proof.** Formulate the following primal and dual linear programming problems:

Minimize $e'z$

subject to

$Ax + z \geq e$, $x \geq 0, z \geq 0$.

Maximize $e'y$

subject to

$A'y \leq 0$, $y \leq e$, $y \geq 0$.

(I) $\Rightarrow$ (II). If (I) does not hold, then $Ax$ has at least one nonpositive component for any $x \geq 0$. So, $z \geq 0$ to satisfy the first constraint of (P). As it is clear that $e'z$ has a lower bound, from Lemma 2 there exists an optimal solution $z^*$ in (P) and $e'z^* > 0$. From Lemma 1 it follows that there exists an optimal solution in (D) such that $e'y^* > 0$, $A'y^* \leq 0$, $y^* \geq 0$. This means that $y^* \geq 0$, and hence (II) holds for $y^*$.

(I) $\Rightarrow$ (II). If (I) holds for $x \geq 0$, then $Ax > 0$ and there is a vector $x^*$ such that $Ax^* = b$, $x^* \geq 0$, where $x^* = \alpha x$ for a sufficiently large number $\alpha$. Thus, $z = 0$, together with $x^*$, is an optimal solution of (P). From Lemma 1 there exists an optimal solution $y^*$ in (D) such that $e'y^* = 0$, $A'y^* \leq 0$, $y^* \geq 0$. Therefore, for any $y$ such that $A'y \leq 0$, $y \geq 0$, there is no $y \geq 0$. For if $y \geq 0$, $0 < e'y \leq e'y^* = 0$, a contradiction. Accordingly, (II) does not hold. Q. E. D.

**Theorem 5 (Gale [2, p. 41]).** Either

(I) $A'x = b$ has a solution $x$,

or

(II) $A'y = 0$, $b'y = 1$ has a solution $y$,

but never both.

**Proof.** We consider the following primal and dual linear programming problems:

Minimize $e'(u + v)$

7 See, for example, Dantzig [1, p. 139], Gale [2, p. 49], Kemp and Kimura [6, p. 4], and Mangasarian [7, p. 35].

8 See also Kemp and Kimura [6, pp. 6–7] and Mangasarian [7, p. 33].
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(P) subject to
\[ Ax + u - v = b, \]
\[ u, v \geq 0. \]
Maximize
\[ b'y. \]

(D) subject to
\[ A'y = 0, \]
\[ y \leq e, \]
\[ -y \leq e. \]

(\(\text{I}\)) \(\Rightarrow\) (\(\text{II}\)). If (I) does not hold, then \(u + v \geq 0\) for any \(x\) in (P). It follows from Lemma 2 that there exists an optimal solution \(u^*, v^*\) in (P) and \(e'(u^* + v^*) > 0\). From Lemma 1 there is \(y^*\) in (D) such that \(b'y^* > 0, A'y^* = 0\). Writing \(b'y^* = \alpha > 0, \gamma = y^*/\alpha\), we see that \(A\gamma = 0, b'\gamma = 1\). Hence, (II) holds for \(\gamma\).

(I) \(\Rightarrow\) (\(\text{II}\)). If (I) holds for \(x^*\), then apparently \(u = v = 0\), together with \(x^*\), is an optimal solution of (P). Therefore, from Lemma 1 there exists an optimal solution \(\gamma^*\) in (D) and \(b'\gamma^* = 0\). Since \(b'y \leq b'y^* = 0\) for any \(y\) such that \(A'y = 0, -e \leq y \leq e\), there does not exist \(y\) such that \(A'y = 0, b'y > 0\). Consequently, (II) does not hold.

Q.E.D.

Theorem 6 (Gale [2, pp. 46-47]). Either

(I) \(Ax \leq b\) has a solution \(x\),
or

(II) \(A'y = 0, b'y = -1, y \geq 0\) has a solution \(y\).
but never both.

Proof. Consider the following primal and dual linear programming problems:

Minimize
\[ e'z \]
\[ \text{subject to } -Ax + z \geq -b, \]
\[ z \geq 0. \]
Maximize
\[ -b'y \]
\[ \text{subject to } A'y = 0, \]
\[ y \leq e, \]
\[ y \geq 0. \]

(\(\text{I}\)) \(\Rightarrow\) (\(\text{II}\)). If (I) does not hold for any \(x\), then at least one component of \(Ax\) is larger than the corresponding component of \(b\), so that we must have \(z \geq 0\) for any \(x\) in (P). From Lemma 2 there exists an optimal solution \(z^* \geq 0\) in (P) and \(e'z^* > 0\). From Lemma 1 (D) has an optimal solution \(y^*\) such that \(b'y^* < 0, A'y^* = 0, y^* \geq 0\). Letting \(b'y^* = -\alpha < 0, \gamma = y^*/\alpha\) for \(\alpha > 0\), we have \(b'\gamma = -1, A\gamma = 0, \gamma \geq 0\). Therefore, (II) is derived.

(I) \(\Rightarrow\) (\(\text{II}\)). If (I) holds for \(x^*\), then \(z = 0\), together with \(x^*\), is an optimal solution of (P). So, there is an optimal solution \(y^*\) in (D) such that \(b'y^* = 0, A'y^* = 0, y^* \geq 0\). Thus, there is no \(y\) such that \(b'y = -1, A'y = 0, y \geq 0\). For otherwise, we see that \(1 = -b'y \leq -b'y^* = 0\), a contradiction. There-

\[\text{See also Kemp and Kimura [6, p. 5] and Mangasarian [7, pp. 33-34].}\]
fore, (II) is obtained. Q. E. D.

Theorem 7 (Gale [2, pp. 47-48]). Either
(I) \( Ax \geq b, \ x \geq 0 \) has a solution \( x \),
or
(II) \( A' y \leq 0, \ b' y > 0, \ y \geq 0 \) has a solution \( y \),
but never both.

Proof. Consider the following primal and dual linear programming problems:

\[
\begin{align*}
\text{Minimize} & \quad e'z \\
\text{subject to} & \quad Ax + z \geq b, \\
& \quad x \geq 0, \ z \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{Maximize} & \quad b' y \\
\text{subject to} & \quad A'y \leq 0, \\
& \quad y \leq e, \\
& \quad y \geq 0.
\end{align*}
\]

(1) \( \Rightarrow \) (2). If (1) is not true, then \( Ax \geq b \) for any \( x \geq 0 \), so that \( z \geq 0 \) must hold in (P). From Lemma 2 (P) has an optimal solution \( z^* \geq 0 \) for which \( e'z^* > 0 \). From Lemma 1 there exists an optimal solution \( y^* \) in (D) such that \( b'y^* > 0, A'y^* \leq 0, y^* \geq 0 \). Hence, (II) holds.

(1) \( \Rightarrow \) (II). If (1) is true, then there exists an optimal solution \( z = 0 \) in (P) and thus from Lemma 1 there exists an optimal solution \( y^* \) in (D) such that \( b'y^* = 0 \). As \( b'y \leq b'y^* = 0 \) for any \( y \) such that \( A'y \leq 0, y \geq 0 \), we conclude that (II) does not hold.

Q. E. D.

Theorem 8 (Mangasarian [7, p. 35]). Either
(I) \( Ax \leq 0, \ x \geq 0 \) has a solution \( x \),
or
(II) \( A'y \geq 0, \ y > 0 \) has a solution \( y \),
but never both.

Proof. Consider the following primal and dual linear programming problems:

\[
\begin{align*}
\text{Minimize} & \quad e'w \\
\text{subject to} & \quad A'y \geq 0, \\
& \quad y + w \geq e, \\
& \quad w \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{Maximize} & \quad e'z \\
\text{subject to} & \quad Ax + z = 0, \\
& \quad z \leq e, \\
& \quad x \geq 0, \ z \geq 0.
\end{align*}
\]

\[10\] See also Kemp and Kimura [6, pp. 5-6] and Mangasarian [7, p. 35].
If (II) does not hold, then \( y > 0 \) for all \( y \) such that \( A'y \geq 0 \). Thus, \( w \geq 0 \) to meet the second constraint of (P). From Lemma 2 there exists an optimal solution \( w^* \geq 0 \) in (P) and \( e'w^* > 0 \). It follows from Lemma 1 that there exists an optimal solution \( x^*, z^* \) in (D) such that \( e'z^* > 0, Ax^* + z^* = 0, x^* \geq 0, z^* \geq 0 \). From this, \( z^* \geq 0, Ax^* \leq 0, x^* \geq 0 \). Therefore, (I) is established.

(II) \( \Rightarrow \) (I). If (II) holds for \( y > 0 \), then \( A'y^* \geq 0, y^* \geq e \), where \( y^* = \alpha y \) for a sufficiently large number \( \alpha \). Thus, \( w = 0 \), together with \( y^* \), is an optimal solution of (P), so that there exists an optimal solution \( x^* \geq 0, z^* \geq 0 \) in (D) such that \( e'z^* = 0, Ax^* + z^* = 0 \). On the other hand, if (I) has a solution \( x^* \geq 0, Ax^* \leq 0, x^* \geq 0 \). Therefore, (II) holds for any \( y \) such that \( A'y \leq 0, y \geq 0 \), and so that \( x^* \geq 0 \) to satisfy the first constraint of (D). Then, \( 0 < e'z \leq e'z^* = 0 \), a contradiction. Therefore, (I) does not occur.

**Theorem 9 (Mangasarian [6, p.36]).** Either

(I) \( Ax > 0, x > 0 \) has a solution \( x \),

or

(II) \( A'y \leq 0, y \geq 0 \) has a solution \( y \),

but never both.

**Proof.** We formulate the following primal and dual linear programming problems:

Minimize \( e'z \)

subject to

\( Ax + z \geq e, \)

\( x \geq 0, z \geq 0. \)

Maximize \( e'y \)

subject to

\( A'y \leq 0, \)

\( y \leq e, \)

\( y \geq 0. \)

(\( \bar{I} \)) \( \Rightarrow \) (II). If (I) is not true, then \( Ax \) has at least one nonpositive component for any \( x > 0 \) and thus to meet the constraint of (P) we must have \( z \geq 0 \). Hence, \( \min e'z > 0 \). It follows from Lemmas 1 and 2 that there exists an optimal solution \( y^* \) in (D) such that \( e'y^* > 0, A'y^* \leq 0, y^* \geq 0 \). This implies that \( y^* \geq 0 \). Therefore, (II) is true for \( y^* \).

(II) \( \Rightarrow \) (\( \bar{I} \)). If (II) holds for \( x > 0 \), then \( Ax^* \geq e \), where \( x^* = \alpha x > 0 \) for a sufficiently large number \( \alpha \). Therefore, \( z = 0 \), together with \( x^* \), is an optimal solution of (P) and \( \min e'z = 0 \). From Lemma 1 there exists an optimal solution \( y^* \) in (D) such that \( e'y^* = 0, A'y^* \leq 0, y^* \geq 0 \). Hence, for any \( y \) such that \( A'y \leq 0, y \geq 0 \), we cannot have \( y \geq 0 \). For if \( y \geq 0 \), then \( 0 < e'y \leq e'y^* = 0 \), a contradiction. Thus, (\( \bar{I} \)) is derived.

**References**


