On Generalization of Farkas-Minkowski's Lemma and Stiemke's Lemma

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Abstract.

This paper discusses the possibiliy of extending the classical Farkas-Minkowski's lemma and Stiemke's lemma to infinite dimensional spaces. We show that the former is a simple consequence of the well-known bipolar theorem in the theory of topological vector space, and the later can be accomplished with some restrictions, which are mild enough to include the spaces of certain economic significance such as $\mathbf{R}\times L_1(M,\boldsymbol{M},\mu)$. Consequently, in the incomplete pure financial markets with dividend D taking values in $L_1(M,\boldsymbol{M},\mu)$, a price-dividend pair $(\pi, D)$is strictly arbitrage-free if and only if there exists a strictly positive continuous linear functional λ on $L_1(M,\boldsymbol{M},\mu)$such that $\pi=\lambda D$.

Introduction.

Since Bewley (1972) established the equilibrium existence theorem for econo-mies with an infinite dimensional commodity space, the theory of general equilibrium in the infinite dimensional context has been extended towards various directions. See Florenzano(1983), Toussaint (1984), Noguchi (1997), et al.

Recently, yet another extension involving economies with incomplete financial markets was attempted by Zhang (1997). To be more specific, Zhang claims to have proven existence of general equilibrium in a two period economy with infinite dimensional commodity space and with incomplete pure financial markets. His proof relies on infinite dimensional versions of Farkas-Minkowski's Lemma and Stiemke's Lemma. Unfortunately, there are some elementary mathematical errors and insufficient arguments in the proofs for establishing the infinite dimensional versions of the two lemmas: for instance, he claims that the relative interior of the standard positive cone of l_1 is non-void, and apparently confuses proper separation with strict separation in the application of the separation theorem.

In this paper, the author shows that a reasonably general infinite dimensional extension of Farkas-Minkowski's Lemma can be established as a simple consequence of the well-known

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bipolar theorem. This direction is by no means new, and indeed appears in Rochafellar(1970)in the finite dimensional context. We also demonstrate that Stiemke's Lemma can be extended in some generality to the spaces of certain economic significance such as $\mathbf{R}\times L_1(M,\boldsymbol{M},\mu)$.

Definitions and Terminology.

For basic definitions and terminology for the theory of topological vector space, we refer the readers to Horvăth (1966), Jameson (1974), Kelley and Namioka (1963), Köthe(1969), and Schaefer (1970). Our definition of topological vector space, t.v.s., in short, does not require Hausdorff property while some authors' do. For a t.v.s. E, E' denotes its topological dual. We follow Schaefer (1970) for the definition of ordered t.v.s., i.e., an ordered t.v.s. E is a t.v.s. with a closed proper cone E_+, called the positive cone of E. Since E_+ is assumed to be closed, the topology of E is automatically Hausdorff(Jameson, 1970, p.80). Let E and F be paired linear spaces with pairing $\langle,\rangle$. For each subset A of E, we define the polar A° of A by $\{f\in F\colon \langle f, x\rangle \geq -1 \text{ for all } x\in A\}$ (Jameson, 1970) so that when E is an ordered t.v.s., we have $E^{\circ}{}_{+}=E'{}_{+}$, where $E'{}_{+}$ is the set of positive continuous linear functionals. Unfortunately, there are several different definitions of polar being used in the current literature, but they all agree on circled subsets, and the difference rarely matters in practice. We rely on the version of bipolar theorem which appeared in Jameson (1970, p.82). $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, where we identify its dual with itself. We write l.c.s. for locally convex topological vector space. $\backslash$ indicates the set theoretic subtraction, *Im* the image of a map, *cl* the topological closure of a set in a topological space, and *int* the topological interior of a set in a topological space.

Farkas-Minkowski's Lemma.

Recall that we have the following classical version of Farkas-Minkowski's Lemma:

Lemma 1 (Farkas-Minkowski).

Let D be a real $n\times m$ matrix. Then a vector $\pi\in\mathbf{R}^n$ satisfies $\pi^t\theta\geq 0$ for all $\theta\in\mathbf{R}^n$ with $D\theta\in\mathbf{R}^m_+$ if and only if there exists a positive vector $\lambda\in\mathbf{R}^m_+$ such that $\pi^t=\lambda^t D$, where t on the shoulders indicates transpose.

We state the following conjecture, which is a simple generalization of the statement in the above lemma:

Conjecture 1.

Let S and T be t.v.s. (not necessary Hausdorff) with topological dual S' and T', respectively. Let $C \subset T$ be a non-empty subset with polar $C^\circ \subset T'$. Let D be a weakly continuous linear map from S into T with dual D'. Note that D' is defined and weakly continuous. For this fact, see, for example, Kelley and Namioka (1963, p.199). *Then $\pi \in S'$ satisfies $\langle \pi, \theta \rangle \geq -1$ for all $\theta \in S$ with $D\theta \in C$ if and only if there exists $\lambda \in C^\circ$ such that $\pi = D'\lambda$.*

Note that the statement in the above conjecture clearly reduces to the classical finite dimensional version when $S = \mathbf{R}^n$, $T = \mathbf{R}^m$, and $C = \mathbf{R}^m_+$.

We can restate the above conjecture in yet another equivalent form, given in terms of polar.

Conjecture 1'.

Under the same hypotheses in Conjecture 1, we have

$$[D^{-1}(C)]^\circ = D'(C^\circ). \qquad (*)$$

We verify the equivalence mentioned above. Observe that the inclusion $D'(C^\circ) \subset [D^{-1}(C)]^\circ$ is a simple consequence of the definition of polar. To see this, let $\pi \in D'(C^\circ)$. Then $\pi = D'\lambda$ for some $\lambda \in C^\circ$.Let $\theta \in D^{-1}(C)$. Then $D\theta \in C$, and hence $-1 \leq \langle \lambda, D\theta \rangle = \langle D'\lambda, \theta \rangle = \langle \pi, \theta \rangle$. Thus $\pi \in [D^{-1}(C)]^\circ$, and $D'(C^\circ) \subset [D^{-1}(C)]^\circ$. Observe that $\pi \in [D^{-1}(C)]^\circ$ if and only if $\langle \pi, \theta \rangle \geq -1$ for all $\theta \in D^{-1}(C)$, i.e., for all θ such that $D\theta \in C$. We also note that $\pi \in D'(C^\circ)$ if and only if there exists $\lambda \in C^\circ$ such that $\pi = D'\lambda$. These establish our claim.

We prove the following lemma:

Lemma 2.

Let $C \subset T$ be a non-empty weakly closed, convex subset containing zero. Then $D^{-1}(C) = [D'(C^\circ)]^\circ$.

Proof for Lemma 2.

$$\begin{aligned} D^{-1}(C) &= D^{-1}(C^{\circ\circ}) \\ &= \{\theta \in S : \langle \lambda, D\theta \rangle \geq -1 \ \forall \lambda \in C^\circ\} \\ &= \{\theta \in S : \langle D'\lambda, \theta \rangle \geq -1 \ \forall \lambda \in C^\circ\} \\ &= [D'(C^\circ)]^\circ. \end{aligned}$$

□

Thus, for $C \subset T$ a non-empty weakly closed, convex subset containing zero, $(*)$ becomes

$$[D'(C^\circ)]^{\circ\circ} = D'(C^\circ).$$

We are ready to prove the following theorem:

Theorem 1 (Generalized Farkas-Minkowski's Lemma).

Let S and T be t.v.s. with dual S' and T', respectively. Let $C \subset T$ be a non-empty weakly closed, convex subset containing zero. Let D be a weakly continuous linear map from S into T with dual D'. If D and D' have a weakly closed range, then $[D'(C^\circ)]^{\circ\circ} = D'(C^\circ)$.

Proof for Theorem 1. Since $D'(C^\circ) \subset S'$ is a non-empty convex subset containing zero, it suffices to show that $D'(C^\circ)$ is weakly closed. To this end, note that since the range of D is assumed to be weakly closed, D' is a weakly relatively open map. For this fact, see, for example, Kelley-Namioka (1963, p.199). Since the range of D' is also assumed to be weakly closed, it is, in fact, a weakly closed map, and our claim follows at once. □

Corollary 1.

Let S and T be l.c.s. with dual S' and T', respectively. Let $C \subset T$ be a non-empty closed, convex subset containing zero. Let D be a weakly continuous linear map from S into T with dual D'. If D and D' have a closed range, then $[D'(C^\circ)]^{\circ\circ} = D'(C^\circ)$.

Proof for Corollary 1. A simple consequence of Kelley and Namioka (1963, p.154) □

Corollary 2.

Let T be an ordered l.c.s. with positive cone T_+. Let D be a linear map from $\mathbf{R}^n$ into T. Then $\pi \in \mathbf{R}^n$ satisfies $\langle \pi, \theta \rangle \geq 0$ for all $\theta \in \mathbf{R}^n$ with $D\theta \in T_+$ if and only if there exists a positive continuous linear functional $\lambda \in T'$ such that $\pi = \lambda D$.

Proof for Corollary 2. Observe that D is weakly continuous (Schaefer, 1970, p.22), and since the weak topology on T is Hausdorff, the range of D, being finite dimensional, is weakly closed (Kelley and Namioka, 1963, p.59). Our claim follows at once from Corollary 1. □

Definition 1.

Let $\pi \in \mathbf{R}^n$ denote the vector of prices of n securities. The securities' price-dividend pair (π, D) is weakly arbitrage-free if any portfolio $\theta \in \mathbf{R}^n$ of securities has positive market value $\langle \pi, \theta \rangle \geq 0$ whenever it has a positive pay-off, or $D\theta \in T_+$.

Corollary 3.

Let T be an ordered l.c.s. with positive cone T_+. Then a price-dividend pair (π,D) is weakly arbitrage-free if and only if there exists a positive continuous linear functional $\lambda \in T'$ such that $\pi = \lambda D$.

Proof for Corollary 3. This follows at once from Corollary 2. □

Stiemke's Lemma.

Definition 2.

Let $\pi \in \mathbf{R}^n$ denote the vector of prices of n securities. The securities' price-dividend pair (π, D) is strictly arbitrage-free if there does not exist a portfolio $\theta \in \mathbf{R}^n$ satisfying $(-\pi\theta, D\theta) \in \mathbf{R}_+ \backslash (0) \times T_+ \backslash (0)$. It follows that (π,D) is strictly arbitrage-free if and only if it is weakly arbitrage-free and any portfolio $\theta \in \mathbf{R}^n$ of securities has strictly positive market value $\langle \pi, \theta \rangle > 0$ whenever it has a strictly positive pay off, or $D\theta \in T_+ \backslash (0)$.

For $T = \mathbf{R}^m$ with the standard positive cone $\mathbf{R}^m_+$, it is well known that (π,D) is strictly arbitrage-free if and only if there exists a strictly positive vector $\lambda \in \mathbf{R}^m$ such that $\pi^t = \lambda^t D$. Thus, one may be tempted to conjecture that an analogous statement holds for an arbitrary l.c.s.. To be more precise, such an statement would be as follows:

Let T be an ordered l.c.s. with positive cone T_+. Then a price-dividend pair (π,D) is strictly arbitrage-free if and only if there exists a strictly positive continuous linear functional $\lambda \in T'$ such that $\pi = \lambda D$.

However, this seemingly plausible statement turns out not to hold for all ordered l.c.s., as the following simple example demonstrates: consider the case where $n=1$, $T=$ the set of real sequences with the product topology, $T_+=$ the standard positive cone, $D\theta = (\theta,0,0,0,\cdots)$, and $\pi = 1 \in \mathbf{R}$. Note that $D\theta \in T_+$ if and only if $\theta \geq 0$ if and only if $\pi\theta \geq 0$, and that $D\theta \in T_+ \backslash (0)$ implies $\theta = \pi\theta > 0$. Thus, (π,D) is clearly strictly arbitrage-free, and should the statement hold in this case, there must be a strictly positive continuous linear functional $\lambda \in T'$ such that $\pi = \lambda D$. However, there does not exist any strictly positive linear functional in the algebraic dual of T (Jameson, 1970, p.34).

This example indicates that the above statement does not hold in full generality and we must restrict the class of ordered l.c.s. T at least to those with positive cone T_+ with a base (Jameson, 1970, p.34), or equivalently, to those whose algebraic dual admits a strictly positive linear functional.

To proceed further, we need the following lemma:

Lemma 3.

Let S be a t.v.s., and T be a locally bounded (Köthe, 1969, p.159), ordered l.c.s. with positive cone T_+. Let D be a linear map from S into T. Let Im D possess the property that every bounded sequence admits a convergent subsequence, and let T_+ be well-based (Jameson, 1970, p. 120). If $Im\ D \cap T_+ = (0)$, there exists a convex cone C such that $T_+ \setminus (0) \subset int\ C$ and $Im\ D \cap int\ C = \phi$.

Proof for Lemma 3. Since T is locally bounded and locally convex, it has a neighborhood filter base at 0 of the form $\beta = \{\sigma U : \sigma > 0\}$, where U is a *bounded convex neighborhood of 0. Let* B be a bounded base for T_+ such that $0 \notin cl\ B$ (Jameson, 1970, p.120). We claim that

$$Im\ D \cap (B + \sigma\ U) = \phi \qquad (**)$$

for some $\sigma > 0$. To this end, suppose there is no $\sigma > 0$ satisfying $(**)$. Then we can choose a decreasing sequence of positive real numbers $\sigma_n \downarrow 0$ in such a way that there is a sequence of points x_n in Im $D \cap (B + \sigma_n\ U)$. Since $x_n \in B + \sigma_n\ U \subset B + \sigma_1\ U$, x_n is a bounded sequence in $Im\ D$. By our hypotheses, x_n admits a subsequence, which is also denoted by x_n, convergent to some $x_n\ Im\ D$. On the other hand, we have $x_n - \sigma_n u_n \in B$ for some sequence $u_n \in U$. Observe that $\sigma_n u_n \to 0$. This fact can be seen as follows: given any neighborhood base $\sigma\ U$ of 0, choose a number n_0 such that for all $n \geq n_0$, we have $\sigma_n\ U \subset \sigma\ U$. Since $\sigma_n u_n \in \sigma_n$ U, the result follows.

Since T_+ is closed and $0 \notin cl\ B$, we have $(x_n - \sigma_n u_n) \to x \in cl\ B \in T_+ \setminus (0)$, and this contradicts the fact that $Im\ D \cap T_+ = (0)$.

Define $C = \bigcup_{u \geq 0} \mu(B + \sigma\ U)$. We show that C has all the desired properties. Since B is, by definition, convex, $B + \sigma\ U$ is also convex. Consequently, C is a convex cone. Let $y \in T_+ \setminus (0)$. Then $y = tb$ for some $t > 0$ and $b \in B$. Observe that $tb \in t(b + \sigma\ int\ U)$, and that $t(b + \sigma\ int\ U)$ is an open neighborhood of $y = tb$ contained in C. Thus, y is an interior point of C. This shows $T_+ \setminus (0) \subset int\ C$.

We next show that $Im\ D \cap int\ C = \phi$. To this end, let $z \in Im\ D \cap C$. If $z \neq 0, z \in Im\ D \cap \mu(B + \sigma\ U)$ for some $\mu > 0$. Then $\mu^{-1} z \in Im\ D \cap (B + \sigma$ U), which contradicts to $(**)$. Thus $Im\ D \cap C = (0)$. Since $0 \notin (B + \sigma\ U)$, 0 is not an interior point of C. Consequently, $Im\ D \cap int\ C = \phi$. □

Theorem 2 (Generalized Stiemke's Lemma).

Let S be a t.v.s., and T be a locally bounded, ordered l.c.s. with positive cone T_+. Let D be a linear *map from S into T. If Im D possesses the property that every bounded sequence admits a convergent subsequence, and if T_+ is well-based, then one and only one of the following statements is correct: (a) There exists a solution $\theta \in S$ to $D\ \theta \in T_+\backslash(0)$.(b) There exists a strictly positive continuous linear functional $\lambda \in T'$ such that $\lambda D=0$.*

Proof for Theorem 2. We assume that both (a) and (b) do not hold, and reach a contradiction. Then, since (a) and (b) cannot hold simultaneously, we establish our claim. Note that we have Im D $\cap$ $T_+=(0)$, and hence, we can apply Lemma 3. By the standard separation argument, there exists $\lambda \in T'$ such that $\lambda(x) \leq$ a for all $x \in Im\ D$, and $a \leq \lambda(x)$ for all $x \in C$. Since $0 \in C$ and $0 \in Im\ D$, we have $a=0$ and $0 \leq \lambda(x)$ for all $x \in C$. We claim that $0 < \lambda(x)$for all $x \in int\ C$. This can be seen as follows: suppose $\lambda(y)=0$ for some $y \in int\ C$. Choose a circled neighborhood V of 0 such that $y+V \subset C$. Since λ maps an open subset onto an open subset of $\mathbf{R}$ (Schaefer, 1970, p.75), λ cannot be identically zero on $y+V$. Thus, there exists a point $z \in V$ such that $\lambda(y+z)\neq 0$. We may assume $\lambda(y+z)=\lambda(z) > 0$. Then $\lambda(y-z)=-\lambda(z) < 0$. However, $y-z \in y-V=y+V \subset C$. This is a contradiction.

We have obtained that $\lambda(x) > 0$ for all x *int C,* and hence for all $x \in T_+\backslash(0)$. This shows that λ is a strictly positive continuous linear functional such that $\lambda(x) \leq 0$ for all $x \in Im\ D$. Since $Im\ D$ is a linear subspace, we have $\lambda(x)=0$ for all $x \in Im\ D$, or λD$=0$. But the last assertion is nothing but (b). □

Corollary 3.

Let T be a locally bounded, ordered l.c.s. with positive cone T_+. Let (π,D) be a price-dividend pair. If T_+ is well-based, (π,D) is strictly arbitrage-free if and only if there exist a strictly positive continuous linear functional $\lambda \in T'$ such that $\pi=\lambda D$.

Proof for Corollary 3. Define $A: \mathbf{R}^n \rightarrow \mathbf{R}\times T$ by $A=(-\pi,D)$. Note that $\mathbf{R}\times T$ is a locally bounded, ordered l.c.s. with positive cone $\mathbf{R}_+\times T_+$. Since $\mathbf{R}\times T$ is Hausdorff, $Im\ A$ is topologically isomorphic to a finite dimensional Euclidean space (Kelley-Namioka, 1963, p.59) and therefore has the property that every bounded sequence admits a convergent subsequence.

We show that $\mathbf{R}_+ T_+$ is well-based. To this end, consider $\tilde{B}=$ the convex hull of $\{(1,0),\{0\}\times B\}$. We claim that $\tilde{B}$ is a bounded base for $\mathbf{R}_+\times T_+$ such that $(0,0) \notin cl\tilde{B}$. Let $(t,x) \in \mathbf{R}_+\backslash(0)\times T_+\backslash(0)$. Then $x=v_x b_x$ for some $b_x \in B$ and a unique positive number v_x. Thus, $(t+v_x)^{-1}(t,x)=(t+v_x)^{-1}t(1,0)+(t+v_x)^{-1}v_x(0,b_x) \in \tilde{B}$, and this shows that $\tilde{B}$ is a base. Since B is bounded, $\tilde{B}$ is clearly bounded. Suppose $(0,0) \in cl\tilde{B}$. We can choose a net $(t_v,x_v) \in \tilde{B}$ such that $(t_v,x_v) \rightarrow (0,0)$. Note that $(t_v,x_v)=\alpha_v(1,0)+(1-\alpha_v)(0,b_v)$ for some $b_v \in B$ and $0 \leq \alpha_v \leq 1$. Hence, we have $b_v=(1-t_v)^{-1}x_v \rightarrow 0$, which is a contradiction since $0 \notin clB$.

We are in a situation, where our version of Stiemke's Lemma can be applied. If (π,D) is strictly arbitrage-free, $A\theta=(-\pi\theta, D\theta) \in \mathbf{R}_+\backslash(0)\times T_+\backslash(0)$ has no solution $\theta \in \mathbf{R}^n$. Thus $\gamma A=0$ must have a solution $\gamma \in (\mathbf{R}\times T)'=\mathbf{R}\times T'$, which is a strictly positive continuous linear functional. Note that $\gamma=\gamma_0+\lambda_0$ for some $\gamma_0 > 0$ and $\lambda_0 \in T'$ a strictly positive continuous linear functional. Thus, we can let $\lambda=\gamma_0^{-1}\lambda_0$ and obtain $\pi=\lambda D$ as desired. The converse is trivial. □

Corollary 4

Let T be a locally bounded, ordered l.c.s. with positive cone T_+. Let (π,D) be a price-dividend pair. If the dual cone $T^{\circ}_+ \subset T'$ admits an interior point with respect to the strong topology $\beta(T', T)$,(π,D) is strictly arbitrage-free if and only if there exist a strictly positive continuous linear functional $\lambda \in T'$ such that $\pi=\lambda D$.

Proof for Corollary 4. A simple consequence of Corollary 3 and Theorem 3.8.4 in Jameson (1970, p.122). □

Corollary 4 holds for $T=L_1(M,\boldsymbol{M},\mu)$ with the standard positive cone, where M is a state space, M is a σ-algebra of subsets of M, and μ is a σ-finite measure on M. This follows from the fact that $\beta(L_\infty(M,\boldsymbol{M},\mu),L_1(M,\boldsymbol{M},\mu))$ coincides with the norm topology and $L_\infty(M,\boldsymbol{M},\mu)_+$ admits an interior point with respect to the norm topology.

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