# N-variable Fubini's Theorem for Young Measures and Iterated Lyapunov's Theorem

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**Abstract**

Recently, Askoura et al. (2013) and Noguchi (2014) obtained non-emptiness results for $\alpha$-cores of $n$-player cooperative games with asymmetric information. Both papers adopt Harsanyi's type description of private information and use Young measure techniques in an essential way but in quite different manners, along with Scarf's (1971) celebrated balancedness argument. Noguchi (2014) presented a partly sketchy proof of an iterated version of Lyapunov's theorem for Young measures, which plays a crucial role in the proof of his main theorem. The aim of this paper is to provide a rigorous mathematical proof of the aforementioned theorem, where the proof relies only on well-known elementary facts about the Lebesgue integral.

**Key words:** $\alpha$-core, asymmetric information, cooperative games, Lyapunov's Theorem, Young Measures

**JEL classification:** C65, D82

## Introduction

Recently, Askoura et al. (2013) [1] and Noguchi (2014) [9] obtained non-emptiness results for $\alpha$-cores of $n$-player cooperative games with asymmetric information. Both papers adopt Harsanyi's type description of private information and use Young measure techniques in an essential way but in quite different manners, along with Scarf's (1971) [10] celebrated balancedness argument. Noguchi (2014) [9] presented a partly sketchy proof of an iterated version of Lyapunov's theorem for Young measures, which plays a crucial role in the proof of his main theorem. The aim of this paper is to provide a rigorous mathematical proof of the aforementioned theorem, where the proof relies only on well-known elementary facts about the Lebesgue integral.

## Preliminaries

In this section we first list several basic properties of abstract Lebesgue integrals, which will be used repeatedly throughout this paper, and then state the standard version of Fubini's theorem in two variables. Moreover, we prove a simple lemma concerning image measures of product measures under a symmetric change of variables. We lastly provide a rigorous inductive proof of an $n$–variable version of Fubini's theorem, which gives rise to a similar result for Young measures.

For an extended real-valued function $f$ we define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$ as in standard text books.

**Lemma 1** Let $(X, \mathcal{X}, \lambda)$ be a measure space and let $f$, $g$ be extended real-valued, $\mathcal{X}$–measurable functions defined on $X$. Suppose that $f = g$ $\lambda$– $a.e.$ and that $\int f d\lambda$ is defined (i.e., at least one of the numbers $\int f^+ d\lambda$ and $\int f^- d\lambda$ is finite). Then $\int g d\lambda$ is defined and $\int g d\lambda = \int f d\lambda$.

**Proof.** See Theorem 12.14 in Hewitt and Stromberg (1965, p. 169) [7]. ■

**Lemma 2** Let $(X, \mathcal{X}, \lambda)$ be a measure space and $f$ an extended real-valued, $\mathcal{X}$–measurable function defined on $X$. If $\int f d\lambda$ is defined and finite (i.e., $\int f^+ d\lambda < \infty$ and $\int f^- d\lambda < \infty$), then $f$ is finite $\lambda$–$a.e.$

**Proof.** See Theorem 12.15 in Hewitt and Stromberg (1965, p. 169) [7]. ■

**Lemma 3** Let $(X, \mathcal{X}, \lambda)$ be a measure space and $f$ a real-valued, $\mathcal{X}$–measurable, $\lambda$–integrable function defined on $X$. Suppose $f = f_1 - f_2$ for nonnegative, real-valued, $\mathcal{X}$–measurable, and $\lambda$–integrable functions $f_1$, $f_2$. Then $\int f d\lambda = \int f_1 d\lambda - \int f_2 d\lambda$.

**Proof.** See Theorem 12.19 in Hewitt and Stromberg (1965, p. 170) [7]. ■

**Theorem 4 (Fubini)** Let $(X, \mathcal{X}, \lambda)$ and $(Y, \mathcal{Y}, \nu)$ be $\sigma$–finite measure spaces and let $(X \times Y, \mathcal{X} \otimes \mathcal{Y}, \lambda \otimes \nu)$ be their product measure space. If $f$ is a nonnegative, extended real-valued, $\mathcal{X} \otimes \mathcal{Y}$–measurable function on $X \times Y$, then:

(i) the function $x \to f(x, y)$ is $\mathcal{X}$–measurable for each $y \in Y$;

(ii) the function $y \to f(x, y)$ is $\mathcal{Y}$–measurable for each $x \in X$;

(iii) the function $x \to \int f(x, y)\, d\nu(y)$ is $\mathcal{X}$–measurable;

(iv) the function $y \to \int f(x, y)\, d\lambda(x)$ is $\mathcal{Y}$–measurable;

(v) $\int f d\lambda \otimes \nu = \int\int f d\lambda d\nu = \int\int f d\nu d\lambda$

**Proof.** See Theorem 21.12 in Hewitt and Stromberg (1965, p. 384) [7]. ■

**Definition 5** Let $(X, \mathcal{X}, \lambda)$ be a measure space and let $L(\lambda)$ denote the set of all extended real-valued functions which are defined $\lambda$–$a.$ $e.$ on $X$ and admit a $\mathcal{X}$–measurable, $\lambda$–integrable extension. For $f \in L(\lambda)$ we define the integral $\int f d\lambda$ to be equal to $\int f_1 d\lambda$, where $f_1$ is a $\mathcal{X}$–measurable, $\lambda$–integrable extension of $f$. Note that in the light of Lemma 1 the integral $\int f d\lambda$ is independent of the choice of an extension $f_1$.

**Lemma 6 (Symmetry Relations for Measures)** For each $i = 1, \cdots, n$ let $(T_i, \mathcal{T}_i, \mu_i)$ be a probability space and let $\left(\prod_{i=1}^{n} T_i, \bigotimes_{i=1}^{n} \mathcal{T}_i, \bigotimes_{i=1}^{n} \mu_i\right)$ be the product probability space of the former $n$ probability spaces. Let $\sigma$ be a permutation on $\{1, \cdots, n\}$ and let $\hat{\sigma} : \prod_{i=1}^{n} T_i \rightarrow \prod_{i=1}^{n} T_{\sigma(i)}$ be the bijection induced by the permutation of coordinates $(t_1, \cdots, t_n) \rightarrow (t_{\sigma(1)}, \cdots, t_{\sigma(n)})$. For brevity we use the symbol $\sigma$ to denote $\hat{\sigma}$ as well. Then the image measure $\sigma_* \bigotimes_{i=1}^{n} \mu_i$ on $\bigotimes_{i=1}^{n} \mathcal{T}_{\sigma(i)}$ induced by $\sigma$ satisfies $\sigma_* \bigotimes_{i=1}^{n} \mu_i = \bigotimes_{i=1}^{n} \mu_{\sigma(i)}$.

**Proof.** Let $\prod_{i=1}^{n} C_{\sigma(i)}$ be a measurable rectangle in $\bigotimes_{i=1}^{n} \mathcal{T}_{\sigma(i)}$. It is known that the family of all such measurable rectangles is a Boolean semialgebra (Neveu, 1965, p. 80, Proposition lll. 3.1 [8]). Hence by the extension theorem for probability measures in Neveu (1965, p. 23 [8]) we only need to confirm that the above two probability measures agree on the family of measurable rectangles. By the definition of induced measures we obtain

$$\begin{aligned}
\sigma_* \bigotimes_{i=1}^{n} \mu_i \left(\prod_{i=1}^{n} C_{\sigma(i)}\right) &= \bigotimes_{i=1}^{n} \mu_i \left[\sigma^{-1}\left(\prod_{i=1}^{n} C_{\sigma(i)}\right)\right] \\
&= \bigotimes_{i=1}^{n} \mu_i \left(\prod_{i=1}^{n} C_i\right) \\
&= \mu_1 (C_1) \cdots \mu_n (C_n) \\
&= \mu_{\sigma(1)} \left(C_{\sigma(1)}\right) \cdots \mu_{\sigma(n)} \left(C_{\sigma(n)}\right) \\
&= \bigotimes_{i=1}^{n} \mu_{\sigma(i)} \left(\prod_{i=1}^{n} C_{\sigma(i)}\right),
\end{aligned}$$

which establishes our claim. ■

The following theorem is a $n$-variable version of celebrated Fubini's theorem. A proof of the regular version (2-variable) can be found in standard text books such as Hewitt and Stromberg (1965, p. 386) [7]. We present below an inductive proof of a $n$-variable version of Fubini's theorem, which is structured so as to provide a further extension to Young measures.

**Theorem 7** Let $n \geq 2$ and let $(T_1, \mathcal{T}_1, \mu_1), \cdots, (T_n, \mathcal{T}_n, \mu_n)$ be $n$ probability spaces. Let $T(n) = \prod_{i=1}^{n} T_i$, $\mathcal{T}(n) = \bigotimes_{i=1}^{n} \mathcal{T}_i$, and $\mu(n) = \bigotimes_{i=1}^{n} \mu_i$ be the obvious $n$-products. Let $f$ be an extended real-valued, $\mathcal{T}(n)$-measurable function on $T$(n). For $i = 1, \cdots, n$, let $\mathcal{N}_i$ denote the set of all $\mu_i$-null sets in $\mathcal{T}_i$. Then if $\int |f| \, d\mu\,(n) < \infty$:

(i) $\exists N_n \in \mathcal{N}_i \forall t_n \notin N_n \exists N_{n-1} \forall t_{n-1} \notin N_{n-1} \cdots \exists N_2 \forall t_2 \notin N_2$

$$f(\cdot, t_2, \cdots, t_n) \in L(\mu_1),$$

$$\int f(t_1, \cdot, t_3, \cdots, t_n)\, d\mu_1 \in L(\mu_2),$$

$$\int \int f(t_1, t_2, \cdot, t_4, \cdots, t_n)\, d\mu_1 d\mu_2 \in L(\mu_3),$$

$$\vdots$$

$$\int \cdots \int f(t_1, \cdots, t_{n-2}, \cdot, t_n)\, d\mu_1 \cdots d\mu_{n-2} \in L\left(\mu_{n-1}\right);$$

(ii) $\int\cdots\int f(t_1,\cdots,t_{n-1},\cdot)\,d\mu_1\cdots d\mu_{n-1}\in L(\mu_n)$ and
$\int f(t_1,\cdots,t_n)\,d\mu(n)=\int\cdots\int f(t_1,\cdots,t_n)\,d\mu_1\cdots d\mu_n$; and
(iii) $\int f(t_1,\cdots,t_n)\,d\mu(n)=\int\cdots\int f(t_1,\cdots,t_n)\,d\mu_{\sigma(1)}\cdots d\mu_{\sigma(n)}$, where $\sigma$ is any permutation on $\{1,\cdots,n\}$.

Proof of (i) and (ii). We proceed by induction on $n$. For $n=2$, we have by Theorem 4 that $\int|f|\,d\mu(2)=\iint|f|\,d\mu_1 d\mu_2<\infty$ and that $\int|f|\,d\mu_1$ is $\mathcal{T}_2$-measurable. Thus by Theorem 2 there exists a $\mu_2$-null set $N_2\in\mathcal{T}_2$ such that $\forall t_2\notin N_2\ \int|f|\,d\mu_1<\infty$, i.e., $\exists N_2\ \forall t_2\notin N_2\ f\in L(\mu_1)$ as desired for (i). We proceed to show that (ii) holds as well. Recalling $|f|=f^++f^-$, we also have $\int f^\pm\,d\mu(2)<\infty$, and again by Theorem 4 we see that $\int f^\pm d\mu_1$ is $\mathcal{T}_2$-measurable. The preceding argument yields $\forall t_2\notin N_2\ \int f^\pm d\mu_1<\infty$. We modify the values of $\int f^\pm d\mu_1$ over $N_2$ by setting them equal to zeros and let $\overline{\int f^\pm d\mu_1}$ denote the modifications. Define $F=\overline{\int f^+d\mu_1}-\overline{\int f^-d\mu_1}$, which is clearly real-valued and $\mathcal{T}_2$-measurable. Since $\forall t_2\notin N_2$, $F=\int f^+d\mu_1-\int f^-\,d\mu_1=\int fd\mu$, it follows that $\forall t_2\notin N_2\ |F|\le\int f^+d\mu_1+\int f^-d\mu_1=\int|f|\,d\mu_1$ and hence $\forall t_2\notin T_2\ |F|\le|f|\,d\mu_1$, which implies $F\in L(\mu_2)$. We then compute

$$\begin{aligned}\iint fd\mu_1 d\mu_2 &= \int Fd\mu_2 \text{ (Definition 5)}\\ &= \int\overline{\int f^+d\mu_1}d\mu_2-\int\overline{\int f^-d\mu_1}d\mu_2 \text{ (Lemma 3)}\\ &= \iint f^+d\mu_1 d\mu_2-\iint f^-d\mu_1 d\mu_2 \text{ (Lemma 1)}\\ &= \int f^+d\mu(2)-\int f^-d\mu(2) \text{ (Theorem 4)}\\ &= \int fd\mu(2),\end{aligned}$$

which establishes (ii). We next prove (i) and (ii) for $n$ assuming that they hold up to $n-1$. To this end, suppose $\int|f|\,d\mu(n)<\infty$. Then applying Theorem 4, we obtain that $\int|f|\,d\mu(n)=\iint|f|\,d\mu(n-1)\,d\mu_n$ and that $\int|f|\,d\mu(n-1)$ is $\mathcal{T}_n$-measurable. Thus by Theorem 2 there exists a $\mu_n$-null set $N_n\in\mathcal{T}_n$ such that $\forall t_n\notin N_n\int|f|\,d\mu(n-1)<\infty$. Combining this with the inductive hypotheses (i) and (ii) for $n-1$, we establish (i) for $n$. We now show that (ii) holds for $n$ as well. Arguing as in the proof for $n=2$, we observe that $\forall t_n\notin N_n\int f^\pm d\mu(n-1)<\infty$ and that $\int f^\pm d\mu(n-1)$ is $\mathcal{T}_n$-measurable. Define $\overline{\int f^\pm d\mu(n-1)}$ to be equal to $\int f^\pm d\mu(n-1)$ outside $N_n$ and to be equal to zero on $N_n$. Then define $F=\overline{\int f^+d\mu(n-1)}-\overline{\int f^-d\mu(n-1)}$, which is clearly real-valued and $\mathcal{T}_n$-measurable. Note that $\forall t_n\notin N_n\ F=\int f^+d\mu(n-1)-\int f^-d\mu(n-1)=\int fd\mu(n-1)$ and that $\forall t_n\notin T_n\ |F|\le\int f^+d\mu(n-1)+\int f^-d\mu(n-1)=\int|f|\,d\mu(n-1)$ and hence that $F\in L(\mu_n)$. We then have by the inductive hypotheses that $\int fd\mu(n-1)=\int\cdots\int fd\mu_1\cdots d\mu_{n-1}\in L(\mu_n)$ as desired for the first part of (ii) for $n$. We next compute

$$\begin{aligned}\iint f d\mu(n-1)\, d\mu_n &= \int F d\mu_n \\ &= \int \overline{\int f^+ d\mu(n-1) d\mu_n} - \int \overline{\int f^- d\mu(n-1) d\mu_n} \\ &= \iint f^+ d\mu(n-1)\, d\mu_n - \iint f^- d\mu(n-1)\, d\mu_n \\ &= \int f^+ d\mu(n) - \int f^- d\mu(n) \\ &= \int f d\mu(n),\end{aligned}$$

which yields by inductive hypothesis (ii) for $n$ that $\int \cdots \int f d\mu_1 \cdots d\mu_{n-1} d\mu_n = \int f d\mu(n)$, as desired for the second part of (ii) for $n$. ■

Proof of (iii). For each permutation $\sigma$ on $\{1, \cdots, n\}$, define $\mu^\sigma(n) = \bigotimes_{i=1}^{n} \mu_{\sigma(i)}$ and $f^\sigma = f \circ \sigma^{-1}$. We then have by Theorem C in Halmos (1974, p. 163) [6] and by Lemma 6 that

$$\begin{aligned}\int |f^\sigma|\, d\mu^\sigma(n) &= \int |f| \circ \sigma^{-1} d\sigma_* \mu(n) \\ &= \int |f| \circ \sigma^{-1} \circ \sigma d\mu(n) \\ &= \int |f|\, d\mu(n),\end{aligned}$$

and hence if $\int |f|\, d\mu(n) < \infty$ then $\int |f^\sigma|\, d\mu^\sigma(n) < \infty$, and it follows from the previous results that

$$\int f^\sigma d\mu^\sigma(n) = \int \cdots \int f d\mu_{\sigma(1)} \cdots d\mu_{\sigma(n)}$$

Applying the earlier argument to $f^{\sigma\pm}$ and $f^\pm$ instead of $|f^\sigma|$ and $|f|$, we obtain $\int f^\sigma d\mu^\sigma(n) = \int f d\mu(n)$, which establishes our claim. ■

## Main results

In this section we first review the definition and some basic properties of Young measures, and then present our main results.

Let $(T, \mathcal{T}, \mu)$ be a probability space, $A$ a metrizable Suslin space (see Castaing et al., 2004, p. 3 for the definition [5]), $\mathcal{B}(A)$ the Borel $\sigma$-algebra of $A$, and $\mathrm{Prob}(A)$ the set of all probability measures on $\mathcal{B}(A)$. A Young measure (or transition probability) from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$ is a function $\delta : T \to \mathrm{Prob}(A)$ with the property that for each $B \in \mathcal{B}(A)$ the function $t \longmapsto \delta(t)(B)$ is $\mathcal{T}$-measurable (see Neveu, 1965, p. 73) [8]; this measurability agrees with the one that $\delta$ must satisfy as a measurable map from $(T, \mathcal{T})$ to $\mathrm{Prob}(A)$ with the topology of weak convergence. A real valued $\mathcal{T} \otimes \mathcal{B}(A)$-measurable function $u(t, a)$ is called a Carathéodory integrand if it is continuous in the second variable a for every $t \in T$ and satisfies $|u(t, a)| \leq \phi(t)$ for some $\mu$-integrable real valued function $\phi$. Let $\mathcal{R}(T, \mathcal{T}, A)$ be the set of all Young measures from $(T, \mathcal{T})$

to $(A, \mathcal{B}(A))$ and let $\mathcal{M}(T, \mathcal{T}, A)$ be the set of all measurable maps from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$. As is well-known, $\mathcal{R}(T, \mathcal{T}, A)$ can be identified with the set of disintegrable probability measures on product space $(T\times A, \mathcal{T}\otimes\mathcal{B}(A))$: Given $\delta \in \mathcal{R}(T, \mathcal{T}, A)$, the rule $E \times B \longmapsto \int_E \delta(t)(B)\, d\mu(t)$, where $E \in \mathcal{T}$ and $B \in \mathcal{B}(A)$, defines a product probability measure $\mu\otimes\delta$ on $\mathcal{T}\otimes\mathcal{B}(A)$. Conversely, any disintegrable probability measure $\pi$ on $\mathcal{T}\otimes\mathcal{B}(A)$ with the marginal on $T$ equal to $\mu$ can be disintegrated as $\pi = \mu\otimes\delta$ for some Young measure $\delta \in (T, \mathcal{T}, A)$. Under the current assumptions on $A$, it is known that every probability measure $\pi$ on $\mathcal{T}\otimes\mathcal{B}(A)$ is disintegrable. See Castaing et al. (2004) [5] for further details in this regard.

**Theorem 8** (Lyapunov's Theorem for Young Measures) Let $(T, \mathcal{T}, \mu)$ be a nonatomic probability space and $A$ a metrizable Suslin space. Let $U = (U_1, \cdots U_m)$ be a $m$-tuple of real-valued, $\mathcal{T}\otimes\mathcal{B}(A)$-measurable functions. If a Young measure $\delta$ from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$ satisfies

$$\int |U|\, d\mu\otimes\delta = \iint |U(t,a)|\, d\delta d\mu < +\infty,$$

where $|U(t, a)| = \left(\sum_{j=1}^{m} |U_j(t,a)|^2\right)^{\frac{1}{2}}$, then there exists a $\mathcal{T}$-measurable map $f\colon T \to A$ such that

$$\int U(t,a)\, d\mu\otimes\delta = \int U(t, f(t))\, d\mu.$$

**Proof.** See Balder (2000, Theorem 5.10, p. 24) [3] and Balder (2008, Theorem 2.1, p. 76) [4]. ■

Let $(T_1, \mathcal{T}_1, \mu_1), \cdots, (T_n, \mathcal{T}_n, \mu_n)$ be $n$ probability spaces and let $A_1, \cdots, A_n$ be $n$ topological spaces with a countable base. Then we have $\mathcal{B}\left(\prod_{i=1}^{n} A_i\right) = \bigotimes_{i=1}^{n} \mathcal{B}(A_i)$ (see, for example, Hewitt and Stromberg, 1965, p. 391 [7]). For $\delta_i \in \mathcal{R}(\mathrm{T}_i, \mathcal{T}_i, A_i)$, $i = 1, \cdots, n$, we can form a product Young measure $\bigotimes_{i=1}^{n} \delta_i \in \mathcal{R}\left(\prod_{i=1}^{n} T_i, \bigotimes_{i=1}^{n} \mathcal{T}_i, \prod_{i=1}^{n} A_i\right)$ by $\left(\bigotimes_{i=1}^{n} \delta_i\right)(t_1, \cdots, t_n) \equiv \bigotimes_{i=1}^{n} \delta_i(t_i)$, where $(t_1, \cdots, t_n) \in \prod_{i=1}^{n} T_i$. Under these assumptions we prove the following lemma:

**Lemma 9** (Symmetry Relations for Young Measures) Define a bijection $\sigma : \prod_{i=1}^{n} (T_i\times A_i) \to \prod_{i=1}^{n} T_i \times \prod_{i=1}^{n} A_i$ by $\sigma(t_1, a_1, \cdots, t_n, a_n) = (t_1, \cdots, t_n, a_1, \cdots, a_n)$. Then $\sigma_* \bigotimes_{i=1}^{n} (\mu_i\otimes\delta_i) = \left(\bigotimes_{i=1}^{n} \mu_i\right) \otimes \left(\bigotimes_{i=1}^{n} \delta_i\right)$.

**Proof.** Let $C\times D \equiv C_1\times \cdots \times C_n \times D_1 \times \cdots \times D_n$ be a measurable rectangle in $\left(\bigotimes_{i=1}^{n} \mathcal{T}_i\right) \otimes \mathcal{B}\left(\prod_{i=1}^{n} A_i\right)$, where $\mathcal{B}\left(\prod_{i=1}^{n} A_i\right) = \bigotimes_{i=1}^{n} \mathcal{B}(A_i)$ as noted above. As in the preceding argument, it suffices to show that the above two measures agree on the measurable rectangles. We obtain

$$\begin{aligned}
\sigma_* \bigotimes_{i=1}^{n} (\mu_i \otimes \delta_i)(C \times D) &= \bigotimes_{i=1}^{n} (\mu_i \otimes \delta_i)\left[\sigma^{-1}(C_1 \times \cdots \times C_n \times D_1 \times \cdots \times D_n)\right] \\
&= \bigotimes_{i=1}^{n} (\mu_i \otimes \delta_i)(C_1 \times D_1 \times \cdots \times C_n \times D_n) \\
&= \int_{C_1} \delta_1 (D_1)\, d\mu_1 \cdots \int_{Cn} \delta_n (D_n)\, d\mu_n \\
&= \int_{C_1 \times \cdots \times C_n} \delta_1 (D_1) \cdots \delta_n (D_n)\, d\left(\bigotimes_{i=1}^{n} \mu_i\right) \\
&= \int_{C} \left(\bigotimes_{i=1}^{n} \delta_i\right)(D)\, d\left(\bigotimes_{i=1}^{n} \mu_i\right) \\
&= \left(\bigotimes_{i=1}^{n} \mu_i\right) \otimes \left(\bigotimes_{i=1}^{n} \delta_i\right)(C \times D),
\end{aligned}$$

which establishes our claim. ■

Recall that a metrizable Suslin space has a countable base for its topology (see Castaing et al., 2004, p. 3 [5]).

**Theorem 10 (Fubini's Theorem for Young Measures)** For each $i=1, \cdots, n$ let $(T_i, \mathcal{T}_i, \mu_i)$ be a probability space and $A_i$ a metrizable Suslin space. Let $T = \prod_{i=1}^{n} T_i$, $A = \prod_{i=1}^{n} A_i$, $\mathcal{T} = \bigotimes_{i=1}^{n} \mathcal{T}_i$, $\mu = \bigotimes_{i=1}^{n} \mu_i$, and $\delta = \bigotimes_{i=1}^{n} \delta_i$ be the obvious $n$-products. Let $H$ be an extended real-valued $\mathcal{T} \otimes \mathcal{B}(A)$−measurable function on $T \times A$ such that $\int |H|\, d\mu \otimes \delta < \infty$. Let $\sigma$ be any permutation on {1, ⋯, n}. Then

$$\begin{aligned}
\text{(i)} \quad \int H d\mu \otimes \delta &= \iint H d\delta d\mu, \\
\text{(ii)} \quad \int H d\mu \otimes \delta &= \int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_n \otimes \delta_n \\
&= \int \cdots \int H d\mu_{\sigma(1)} \otimes \delta_{\sigma(1)} \cdots d\mu_{\sigma(n)} \otimes \delta_{\sigma(n)}, \text{ and} \\
\text{(iii)} \quad \int H d\mu \otimes \delta &= \int \cdots \int H d\delta_1 d\mu_1 \cdots d\delta_n d\mu_n \\
&= \int \cdots \int H d\delta_{\sigma(1)} d\mu_{\sigma(1)} \cdots d\delta_{\sigma(n)} d\mu_{\sigma(n)},
\end{aligned}$$

where the $n$ iterated integrals appeared on the right hand sides can be defined successively.

**Proof of (i).** Recall that we defined $\int H d\mu \otimes \delta \equiv \int H^+ d\mu \otimes \delta - \int H^- d\mu \otimes \delta$. We assume that the assertions hold for nonnegative extended real-valued functions (see, for example, Theorem 2.1. in Neveu, 1965, p. 73 [8]). Since $\int |H|\, d\mu \otimes \delta = \iint |H|\, d\delta\, d\mu < \infty$ and since by Theorem 2.1. in Neveu (1965, p. 73 [8]) $\int |H|\, d\delta$ is $\mathcal{T}$-measurable, Theorem 2 implies that there exists a $\mu$-null set $N \in \mathcal{T}$ such that $\forall t \notin N \int |H|\, d\delta < \infty$. On the other hand, applying the preceding argument to $H^{\pm}$ instead of $|H|$, we see that $\int H^{\pm} d\delta$ is $\mathcal{T}$−measurable and $\forall t \notin N \int H^{\pm} d\delta < \infty$. Let $\overline{\int H^{\pm} d\delta}$ denote the modification of $\int H^{\pm} d\delta$ obtained by replacing the values over N by zeros. Define $F = \overline{\int H^+ d\delta} - \overline{\int H^- d\delta}$, which is clearly real-valued and $\mathcal{T}$−measurable. Since $\forall t \notin N$, $F = \int H^+ d\delta - \int H^- d\delta = \int H d\delta$, it follows that $\forall t \notin T\ |F| \leq \int |H|\, d\delta$, which implies that $F$ is $\mu$-

integrable. By Definition 5 and Lemma 3, we obtain

$$\begin{aligned}\iint Hd\delta d\mu &= \int Fd\mu \\ &= \int\overline{\int H^{+}d\delta d\mu} - \int\overline{\int H^{-}d\delta d\mu} \\ &= \iint H^{+}d\delta d\mu - \iint H^{-}d\delta d\mu \\ &= \int H^{+}d\mu\otimes\delta - \int H^{-}d\mu\otimes\delta \\ &= \int Hd\mu\otimes\delta,\end{aligned}$$

as desired. ■

Remark 11 Let $H\in L(\mu\otimes\delta)$ and let $H_1$ be a measurable extension of $H$. Then we have $\int|H_1|d\mu\otimes\delta<\infty$ and by the previous result we obtain $\int H_1 d\mu\otimes\delta=\iint H_1 d\delta d\mu$. Let $H_2$ be another measurable extension of $H$. Then we have, as above, $\int H_2 d\mu\otimes\delta=\iint H_2 d\delta d\mu$ and consequently $\iint H_1 d\delta d\mu=\iint H_2 d\delta d\mu$ since $\int H_1 d\mu\otimes\delta=\int H_2 d\mu\otimes\delta$. Thus we may define

$$\iint Hd\delta d\mu := \iint H_1 d\delta d\mu,$$

where $H_1$ is any measurable extension of $H$. We can now assert that for $H\in L(\mu\otimes\delta)$,

$$\int Hd\mu\otimes\delta = \iint Hd\delta d\mu$$

holds as well.

Proof of (ii). Consider the permutation of coordinates $\sigma:\prod_{i\in N}T_i\times A_i\to T\times A$ defined by $\sigma(t_1, a_1, \cdots, t_n, a_n)=(t_1, \cdots, t_n, a_1, \cdots, a_n)$. Then $H\circ\sigma(t_1, a_1, \cdots, t_n, a_n)=H(t, a)$. Note that by Lemma 9 we have

$$\begin{aligned}\int H\circ\sigma d\left(\bigotimes_{i=1}^{n}\mu_i\otimes\delta_i\right) &= \int Hd\left(\sigma_*\bigotimes_{i=1}^{n}\mu_i\otimes\delta_i\right) \\ &= \int Hd\mu\otimes\delta.\end{aligned}$$

By Theorem 7, we obtain

$$\begin{aligned}\int Hd\mu\otimes\delta &= \int\cdots\int H\circ\sigma d\mu_1\otimes\delta_1\cdots d\mu_n\otimes\delta_n \\ &= \int\cdots\int Hd\mu_1\otimes\delta_1\cdots d\mu_n\otimes\delta_n,\end{aligned}$$

where the iterated integrals appeared above can be defined successively. The rest of the assertion follows from Theorem 7 (iii). ■

Proof of (iii). Throughout this part of the proof, for each $i=1, \cdots, n$, $N_i\in\mathcal{T}_i\otimes\mathcal{B}(A_i)$ denotes a $\mu_i\otimes\delta_i$-null set. We proceed by induction on $n$. To this end we set up the following inductive hypotheses for $n\geq 2$.

(a) $\exists N_n \forall (t_n, a_n) \notin N_n \exists N_{n-1} \forall (t_{n-1}, a_{n-1}) \notin N_{n-1} \cdots \exists N_2 \forall (t_2, a_2) \notin N_2$

$H \in L(\mu_1 \otimes \delta_1)$

and $\int H d\mu_1 \otimes \delta_1 = \iint H d\delta_1 d\mu_1$,

$\int H d\mu_1 \otimes \delta_1 \in L(\mu_2 \otimes \delta_2)$

and $\iint H d\mu_1 \otimes \delta_1 d\mu_2 \otimes \delta_2 = \int \cdots \int H d\delta_1 d\mu_1 d\delta_2 d\mu_2$,

$\vdots$

$\int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_{n-2} \otimes \delta_{n-2} \in L(\mu_{n-1} \otimes \delta_{n-1})$

and $\int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1} = \int \cdots \int H d\delta_1 d\mu_1 \cdots d\delta_{n-1} d\mu_{n-1}$

(b)

$$\int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1} \in L(\mu_n \otimes \delta_n)$$

$$\text{and} \int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_n \otimes \delta_n = \int \cdots \int H d\delta_1 d\mu_1 \cdots d\delta_n d\mu_n.$$

In what follows, let $T(n) = \prod_{i=1}^n T_i$, $A(n) = \prod_{i=1}^n A_i$, $\mathcal{T}(n) = \bigotimes_{i=1}^n \mathcal{T}_i$, $\mu(n) = \bigotimes_{i=1}^n \mu_i$, and $\delta(n) = \bigotimes_{i=1}^n \delta_i$. For $n=2$, suppose $\infty > \int |H| d\mu(2) \otimes \delta(2) = \int |H| d(\mu_1 \otimes \delta_1) \otimes (\mu_2 \otimes \delta_2)$, where the last equality follows from Lemma 9. Then by Theorem 7 $\exists N_2 \forall (t_2, a_2) \notin N_2 \int |H| d(\mu_1 \otimes \delta_1) < \infty$, which implies by (i) above that $\exists N_2 \forall (t_2, a_2) \notin N_2 \int H d\mu_1 \otimes \delta_1 = \iint H d\delta_1 d\mu_1$. We also obtain $\int H d\mu_1 \otimes \delta_1 \in L(\mu_2 \otimes \delta_2)$, which implies $\iint H d\delta_1 d\mu_1 \in L(\mu_2 \otimes \delta_2)$. Thus by Remark 11 $\iint H d\mu_1 \otimes \delta_1 d\mu_2 \otimes \delta_2 = \int \cdots \int H d\delta_1 d\mu_1 d\mu_2 \otimes \delta_2 = \int \cdots \int H d\delta_1 d\mu_1 d\delta_2 d\mu_2$ as desired.

Suppose that the assertion holds up to $n-1$ and that $\infty > \int |H| d\mu(n) \otimes \delta(n)$. We then have by Lemma 9 that

$$\begin{aligned}\infty > \int |H| d\mu(n) \otimes \delta(n) &= \int |H| d\mu(n-1) \otimes \mu_n \otimes \delta(n-1) \otimes \delta_n \\ &= \iint |H| d\mu(n-1) \otimes \delta(n-1) d\mu_n \otimes \delta_n.\end{aligned}$$

It follows that there exists a $\mu_n \otimes \delta n-$null set $N_n$ such that $\forall (t_n, a_n) \notin N_n \int |H| d\mu(n-1) \otimes \delta(n-1) < \infty$. Then by the inductive hypotheses $\forall (t_n, a_n) \notin N_n$ and both (a) and (b) hold. It remains to show that

$$\int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1} \in L(\mu_n \otimes \delta_n)$$

$$\text{and} \int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_n \otimes \delta_n = \int \cdots \int H d\delta_1 d\mu_1 \cdots d\delta_n d\mu_n.$$

To this end, observe that $\forall (t_n, a_n) \notin N_n$

$$\int H d\mu(n-1) \otimes \delta(n-1) = \int \cdots \int H d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1},$$

which follows from part (ii) above. We then have

$$\begin{aligned} & \int\left|\int\cdots\int H d\mu_1\otimes\delta_1\cdots d\mu_{n-1}\otimes\delta_{n-1}\right| d\mu_n\otimes\delta_n \\ = & \int\left|\int H d\mu(n-1)\otimes\delta(n-1)\right| d\mu_n\otimes\delta_n \\ \leq & \int\int |H|\, d\mu(n-1)\otimes\delta(n-1)\, d\mu_n\otimes\delta_n \\ < & \infty, \end{aligned}$$

which verifies the first part of the assertion. Observe from the last line in (a) and the first line in (b) that

$$\int\cdots\int H d\delta_1 d\mu_1\cdots d\delta_{n-1}d\mu_{n-1}\in L(\mu_n\otimes\delta_n).$$

We then compute

$$\begin{aligned} & \int\cdots\int H d\mu_1\otimes\delta_1\cdots d\mu_{n-1}\otimes\delta_{n-1}d\mu_n\otimes\delta_n \\ = & \int\left(\int\cdots\int H d\delta_1 d\mu_1\cdots d\delta_{n-1}d\mu_{n-1}\right) d\mu_n\otimes\delta_n \\ = & \int\cdots\int H d\delta_1 d\mu_1\cdots d\delta_{n-1}d\mu_{n-1}d\delta_n d\mu_n, \end{aligned}$$

where the last equality follows from Remark 11. This completes the proof of the second part of the assertion. ■

Theorem 12 (Iterated Lyapunov's Theorem for Young Measures) For each $i$=1, ⋯, $n$ let $(T_i, \mathcal{T}_i, \mu_i)$ be a nonatomic probability

space and let $A_i$ be a metrizable Suslin space. Suppose $u: T\times A\to\mathbb{R}$ is a Carathéodory integrand, where vert $|u(t,\cdot)|\leq\phi(t)\ \forall t\in T$ for some nonnegative, $\mu$−integrable real-valued function $\phi$. Let $\mu(n)=\bigotimes_{i=1}^{n}\mu_i$ and $\delta(n)=\bigotimes_{i=1}^{n}\delta_i$. Then there exists a measurable map $f_i: T_i\to A_i$ for each $i$= 1, ⋯, $n$ such that

$$\int u(t_1,\cdots t_n,a_1,\cdots,a_n)\, d\mu(n)\otimes\delta(n)=\int u(t_1,\cdots t_n,f_1(t_1),\cdots,f_n(t_n))\, d\mu(n).$$

Proof. Throughout the proof, $N_i$ denotes a $\mu_i$−null set. We let $T(n)=\prod_{i=1}^{n}T_i$, $t(n)=(t_1,\cdots,t_n)$, $A(n)=\prod_{i=1}^{n}A_i$, $a(n)=(a_1,\cdots,a_n)$, and $\mathcal{T}(n)=\bigotimes_{i=1}^{n}\mathcal{T}_i$. We proceed by induction. For $n$=1, since $\int|u|\, d\mu_1\otimes\delta_1\leq\infty\ \phi\ d\mu_1<\infty$, we have by Theorem 8 that there is a measurable map $f_1: T_1\to A$ satisfying $\int u(t_1,a_1)\, d\mu_1\otimes\delta_1=\int u(t_1,f_1(t_1))\, d\mu_1$. Suppose the result holds up to $n-1$. Note that since

$$\int\int\phi d\mu(n-1)\, d\mu_n=\int\phi d\mu(n)<\infty,$$

$\exists N_n\forall t_n\notin N_n\int\phi\, d\mu(n-1)<\infty$. Then we have $\forall(t_n,a_n)\in N_n^c\times A_n$

$$\int|u|\, d\mu(n-1)\otimes\delta(n-1)\leq\int\phi d\mu(n-1)<\infty,$$

and hence by Theorem 10 that $\forall\,(t_n, a_n) \in N_n^c \times A_n$

$$\int u d\mu\,(n-1) \otimes \delta\,(n-1) = \int \cdots \int u d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1}.$$

Observe that $N_n^c \times A_n$ is a $\mu_n \otimes \delta_n$–full set, which can be verified as follows:

$$\begin{aligned}\mu_n \otimes \delta_n\,(N_n^c \times A_n) &= \int_{N_n^c} \delta_n\,(A_n)\,d\mu_n \\ &= \mu_n\,(N_n^c) \\ &= 1.\end{aligned}$$

Thus $\int u d\mu\,(n-1) \otimes \delta\,(n-1) = \int \cdots \int u d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1}$ is defined $\mu_n \otimes \delta_n-$a.e. on $T_n \times A_n$. Define

$$F\,(t_n, a_n) = \begin{cases} \int u d\mu\,(n-1) \otimes \delta\,(n-1) \\ \qquad = \int \cdots \int u d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1} & \text{on } N_n^c \times A_n \\ \qquad\qquad 0 & \text{on } N_n \times A_n. \end{cases}$$

Since by Theorem 4 $\int u^{\pm} d\mu\,(n-1) \otimes \delta\,(n-1)$ is $\mathcal{T}_n \otimes \mathcal{B}(A_n)$ −measurable and since $\int u^{\pm} d\mu\,(n-1) \otimes \delta\,(n-1) < \infty$ on $N_n^c \times A_n$, $F$ is real-valued and $\mathcal{T}_n \otimes \mathcal{B}(A_n)$ −measurable. Furthermore, since

$$\left| \int u d\mu\,(n-1) \otimes \delta\,(n-1) \right| \leq \int |u|\,d\mu\,(n-1) \otimes \delta\,(n-1)$$

on $N_n^c \times A_n$, which implies $\forall (t_n, a_n) \in T_n \times A_n$ $|F| \leq \int |u|\,d\mu\,(n-1) \otimes \delta\,(n-1)$, and since $\iint |u|\,d\mu\,(n-1) \otimes \delta\,(n-1)\,d\mu_n \otimes \delta_n \leq \int \phi\,d\mu(n) < \infty$, $F$ is $\mu_n \otimes \delta_n-$integrable as well. Applying Theorem 8, we obtain a measurable map $f_n : T_n \to A_n$ such that

$$\int F\,(t_n, a_n)\,d\mu_n \otimes \delta_n = \int F\,(t_n, f_n\,(t_n))\,d\mu_n.$$

It is important to observe that

$$\int u\,(t\,(n-1)\,, t_n, a\,(n-1)\,, f_n\,(t_n))\,d\mu\,(n-1) \otimes \delta\,(n-1) = F\,(t_n, f_n\,(t_n))$$

holds on $N_n^c$, where the LHS is defined $\mu_n - a.e.$ on $T_n$ whereas the RHS is a measurable extension of the former. Then by Definition 5, we obtain

$$\begin{aligned} &\int u d\mu\,(n) \otimes \delta\,(n) \\ =& \int \left[ \int \cdots \int u d\mu_1 \otimes \delta_1 \cdots d\mu_{n-1} \otimes \delta_{n-1} \right] d\mu_n \otimes \delta_n \\ =& \int F\,(t_n, a_n)\,d\mu_n \otimes \delta_n \\ =& \int F\,(t_n, f_n\,(t_n))\,d\mu_n \\ =& \int \left[ \int u\,(t\,(n-1)\,, t_n, a\,(n-1)\,, f_n\,(t_n))\,d\mu\,(n-1) \otimes \delta\,(n-1) \right] d\mu_n. \end{aligned}$$

Since

$$\begin{aligned} & \int\int |u(t(n-1), t_n, a(n-1), f_n(t_n))|\, d\mu(n-1)\otimes\delta(n-1)\, d\mu_n \\ \leq & \int\int \phi d\mu(n-1)\otimes\delta(n-1)\, d\mu_n \\ = & \int \phi d\mu < \infty, \end{aligned}$$

we obtain by Fubini's theorem that

$$\begin{aligned} & \int u d\mu(n)\otimes\delta(n) \\ = & \int\left[\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu(n-1)\otimes\delta(n-1)\right] d\mu_n \\ = & \int\left[\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n\right] d\mu(n-1)\otimes\delta(n-1). \end{aligned}$$

Note that $\int u(t(n-1), t_n, a(n-1), f_n(t_n))\ d\mu_n$ is defined $\mu(n-1)\otimes\delta(n-1)-a.e.$ on $T(n-1)\times A(n-1)$; we next construct a measurable extension $G(t(n-1), a(n-1))$ which is integrable so that one can apply the inductive hypothesis to obtain measurable functions $f_1, \cdots, f_{n-1}$ to "purify" $\delta(n-1)$. To this end, observe that

$$\int |u(t(n-1), t_n, a(n-1), f_n(t_n))|\, d\mu_n \leq \int \phi d\mu_n$$

for all $(t(n-1), a(n-1))$. Since $\int\int \phi d\mu_n d\mu(n-1) = \int \phi d\mu(n) < \infty$, there exists a $\mu(n-1)-$null set $N(n-1)\in\mathcal{T}(n-1)$ such that $\forall t(n-1)\notin N(n-1) \int \phi d\mu_n < \infty$ and hence $\forall(t(n-1), a(n-1))\in N(n-1)^c\times A(n-1)$

$$\int |u(t(n-1), t_n, a(n-1), f_n(t_n))|\, d\mu_n < \infty.$$

Thus $\forall(t(n-1), a(n-1))\in N(n-1)^c\times A(n-1)$

$$\begin{aligned} & \left|\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n\right| \\ \leq & \int |u(t(n-1), t_n, a(n-1), f_n(t_n))|\, d\mu_n \\ \leq & \int \phi d\mu_n < \infty. \end{aligned}$$

Moreover, note that since

$$\begin{aligned} & \mu(n-1)\otimes\delta(n-1)(N(n-1)^c\times A(n-1)) \\ = & \int_{N(n-1)^c} \delta(n-1)(A(n-1))\, d\mu(n-1) \\ = & \mu(n-1)(N(n-1)^c) \\ = & 0, \end{aligned}$$

$N(n-1)^c\times A(n-1)$ is a $\mu(n-1)\otimes\delta(n-1)-$full set and hence $\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n$ is defined $\mu(n-1)\otimes\delta(n-1)-a.e.$ on $T(n-1)\times A(n-1)$. Define

$$G(t(n-1), a(n-1)) = \begin{cases} \int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n & \text{on } N(n-1)^c \times A(n-1) \\ 0 & \text{on } N(n-1) \times A(n-1) \end{cases}.$$

As in the previous argument for $F$, it is easily seen that $G(t(n-1), a(n-1))$ is $\mathcal{T}(n-1) \otimes \mathcal{B}(A(n-1))$ − measurable, real-valued, and $\mu(n-1) \otimes \delta(n-1)$ − integrable. Thus by Definition 5,

$$\begin{aligned} &\int G(t(n-1), a(n-1))\, d\mu(n-1) \otimes \delta(n-1) \\ = &\iint u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n d\mu(n-1) \otimes \delta(n-1) \\ = &\int u d\mu(n) \otimes \delta(n). \end{aligned}$$

Recall that

$$\begin{aligned} |G(t(n-1), a(n-1))| &\leq \left| \int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n \right| \\ &\leq \int \phi d\mu_n \end{aligned}$$

on $N(n-1)^c \times A(n-1)$, where $\int \phi d\mu_n$ is defined on $N(n-1)^c$. Define a measurable extension of $\int \phi d\mu_n$ by

$$\widetilde{\phi}(t(n-1)) = \begin{cases} \int \phi(t(n-1), t_n)\, d\mu_n & \text{on } N(n-1)^c \\ 0 & \text{on } N(n-1) \end{cases}.$$

We then have $\forall (t(n-1), a(n-1)) \in T(n-1) \times A(n-1)\ |G(t(n-1), a(n-1))| \leq \widetilde{\phi}(t(n-1))$. We show that $\widetilde{\phi}(t(n-1))$ is $\mu(n-1) \otimes \delta(n-1)$ − integrable to ensure that $G$ is a Carathéodory integrand on $T(n-1) \times A(n-1)$. To this end, note that by Definition 5

$$\begin{aligned} \int \widetilde{\phi}(t(n-1))\, d\mu(n-1) &= \iint \phi(t(n-1), t_n)\, d\mu_n d\mu(n-1) \\ &= \int \phi d\mu(n) < \infty, \end{aligned}$$

which establishes the claim.

By the inductive hypothesis, there exist measurable functions $f_i: T_i \rightarrow A_i$, $i = 1, \cdots, n-1$, such that

$$\begin{aligned} &\int G(t(n-1), a(n-1))\, d\mu(n-1) \otimes \delta(n-1) \\ = &\int G(t(n-1), f(n-1)(t(n-1)))\, d\mu(n-1), \end{aligned}$$

where $f(n-1) = (f_1, \cdots, f_{n-1})$ and $f(n-1)(t(n-1)) = (f_1(t_1), \cdots, f_{n-1}(t_{n-1}))$. Observe that

$$G(t(n-1), f(n-1)(t(n-1))) = \begin{cases} \int u(t(n-1), t_n, f(n-1)(t(n-1)), f_n(t_n))\, d\mu_n & \text{on } N(n-1)^c \\ 0 & \text{on } N(n-1) \end{cases}$$

and that $G(t(n-1), f(n-1)(t(n-1)))$ is a measurable extension of $\int u(t(n-1), t_n, f(n-1)$

$(t\ (n-1)),\ f_n(t_n))\,d\mu_n$. Thus by Definition 5 we obtain

$$\begin{aligned}
&\int u d\mu(n) \otimes \delta(n) \\
=\ &\int \left[\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu_n\right] d\mu(n-1) \otimes \delta(n-1) \\
=\ &\int G(t(n-1), a(n-1))\, d\mu(n-1) \otimes \delta(n-1) \\
=\ &\int G(t(n-1), f(n-1)(t(n-1)))\, d\mu(n-1) \\
=\ &\iint u(t(n-1), t_n, f(n-1)(t(n-1)), f_n(t_n))\, d\mu_n d\mu(n-1) \\
=\ &\iint u(t(n-1), t_n, f(n-1)(t(n-1)), f_n(t_n))\, d\mu(n-1)\, d\mu_n \\
=\ &\int u(t_1, \cdots t_n, f_1(t_1), \cdots, f_n(t_n))\, d\mu(n),
\end{aligned}$$

where the step from the fifth line to the sixth line follows since $\int |u(t(n-1), t_n, f(n-1)(t(n-1)), f_n(t_n))|\ d\mu(n) \leq \int \phi d\mu(n) < \infty$. This completes the proof. ■

Remark 13 It is important to note that the usual condition

$$\int |u|\, d\mu(n) \otimes \delta(n) < \infty$$

does not generally guarantee that our inductive proof method works. For example, $F(t_n, a_n)$ may be a measurable extension of

$$\int u(t(n-1), t_n, a(n-1), a_n)\, d\mu(n-1) \otimes \delta(n-1)$$

while $F(t_n, f_n(t_n))$ may not be a measurable extension of

$$\int u(t(n-1), t_n, a(n-1), f_n(t_n))\, d\mu(n-1) \otimes \delta(n-1).$$

This problem can be avoided if the former function is defined on a $\mu_n \otimes \delta_n$−full set $N_n^c \times A_n$ as shown in the preceding paragraph. Such an arrangement is possible if u is a Carathéodory integrand.

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