Negligibility of Dictatorial Market Mechanisms

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Abstract

We demonstrate that in the totality of suitably chosen (complete, acyclic) Arrovian collective choice rules, the ones with dictatorship form a negligible subset provided that there are more than two significant groups of individuals and that the number of available alternatives is infinite. Regarding social choice theory, we derive under these conditions that among all possible finitely complete Arrovian social choice functions satisfying certain consistency conditions, the ones with dictatorship are negligible and unstable. We construct as an example of such social choice functions a market mechanism which chooses socially most desired prices, and argue that a market in which a particular trader can always correctly predict prices exists only rarely and unstably.

Introduction

A social welfare function F is a function that assigns a unique social preference ordering $\succsim$ to each preference profile $\{\succsim_i\}$, a combination of individual preference orderings $\succsim_i$ with one $\succsim_i$ for each i, and is relevant to social choice theory in as much as F generates a social choice function $C_{\{\cdot\}}$, a function that chooses most socially desired alternatives among currently feasible alternatives in a given choice situation. In words, a social welfare function aggregates individual values or tastes to give rise to a social preference ordering and thereby enables society to choose best alternatives from each admissible agenda. More than half a century ago, Arrow (1963) cogently argued that every social welfare function, though merely satisfying a set of apparently innocuous conditions, admits a dictator, or to be more precise, a collectively rational aggregation of individual preference orderings becomes impossible without dictatorship if we impose three natural conditions: universal domain (the domain of F contains every logically possible preference profile); independence of irrelevant alternatives (for each pair of alternatives x, y, the social preference over x, y depends on individual preferences only over x, y); and unanimity (for each pair of alternatives x, y, if every

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individual strictly prefers x to y, so does society).

While many have attempted to circumvent the above inconsistency result of Arrow, by weakening, for example, some of the conditions deemed plausible by Arrow and others, no work seems to have appeared in the literature to date which provides insight into how likely or unlikely those unfavorable social welfare functions involving dictators are to actually emerge (Noguchi, 2011). Mathematically speaking, such a discussion would inevitably involve determining the totality of collective choice rules (preference aggregation rules like social choice functions but with no specification of a particular collective rationality such as transitivity, etc.) that are admissible in a given sense and of how large a region the ones with dictatorship occupy in the totality of all admissible ones.

In this paper, we demonstrate that in the totality of Arrovian (i.e., satisfying universal domain, independence of irrelevant alternatives, and unanimity) collective choice rules with a level of collective rationality (complete, acyclic) that is exactly sufficient to generate the so-called "finitely complete" social choice functions (Sen, 1986), the ones with dictatorship form a negligible subset provided that there are more than two significant groups of individuals and that the number of available alternatives is infinite. Actually, we can say more regarding the relevance of our results to social choice theory : While the social choice functions generated by the above class of collective choice rules are known to satisfy the Chernoff condition (a contraction consistency) and the generalized Condorcet property (an expansion consistency), any finitely complete Arrovian social choice function satisfying these conditions is known to be rationalized, i.e. generated, by a collective choice rule belonging to the above class (i.e., by its own revealed preference relation). On account of the above remarks, we may conclude that in societies with more than two persons and with infinitely many alternatives, among all possible finitely complete Arrovian social choice functions satisfying the Chernoff condition and the generalized Condorcet property, the ones with dictatorship are negligible and hence rarely emerge.

Although the Pareto rule, being transitive but not complete, does not give rise to a social choice function which is finitely complete Arrovian, satisfying the Chernoff condition and the generalized Condorcet property, the so-called Pareto extension rule does, since it is quasi-transitive (but not transitive) and complete. Observe that the Pareto extension rule, being quasi-transitive and complete, is acyclic and complete and in fact, is a member of the class of social choice functions that we are currently focusing on. Another interesting attribute of the Pareto extension rule is that it admits no dictatorship since the rule transforms unequivocal but dissenting individual preferences into an indifferent social preference. We demonstrate in conclusion that a simple yet realistic asset market mechanism can induce the Pareto extension rule and hence a market price choice mechanism without dictatorship, which in our view is an archetypal attribute of all such price choice mechanisms.

As well known in the literature, the genericity argument, the line of argument employed in this paper, proved to be a powerful tool for showing that a set of certain undesirable outcomes or situations is negligible. For example, Debreu (1970) and many others made use of the genericity argument to prove that almost all economies are regular and hence the equilibria of which are locally unique. Husseini, Lasry, and Magill (1990) also used the genericity argument to establish the generic existence of equilibrium for the general equilibrium model with incomplete market (GEI model). In a similar vein, the present paper essentially, though not exactly, asserts that almost all market mechanisms that are capable of social choice are non-dictatorial in the sense that a trader who can always correctly predict the prices is non-existent, and hence claims that reasonable market mechanisms admitting as traders those legendary infallible hedge fund managers are negligible and hence unlikely to materialize. This assertion can be viewed as a variant of the efficient market hypothesis about the predictability of market prices in a totally different flavor, obtained by an application of Arrow's theorem and social choice theory.

Arrovian collective choice rules *F*

In what follows, we use the following standard notation of mathematical logic: $\forall$ the universal quantifier, $\exists$ the existential quantifier, $\wedge$ conjunction, $\vee$ disjunction, $\Rightarrow$ conditional "if then", and $\Leftrightarrow$ conditional "if and only if (iff for short)".

For simplicity of exposition, we only consider 2-person society $T=\{1, 2\}$, where 1 and 2 represent the first and the second individual, respectively. A **collective choice rule** is simply a rule that transforms each preference profile (i.e., tuples of individual preference relations) $(\succsim_1, \succsim_2)$ into a unique social preference relation denoted by $\succsim$. In the sequel we assume that preference relations are defined on a set of **alternatives** $A=\{x, y, z, \ldots\}$ and that each individual preference is an **ordering**, that is, a complete $(\forall x \forall y\, x \succsim_i y \vee y \succsim_i x$ holds) and transitive $(\forall x \forall y \forall z\, x \succsim_i y \wedge y \succsim_i z \Rightarrow x \succsim_i z)$ preference relation. We call a collective choice rule F **Arrovian** if it satisfies the following:

(**Universal Domain, UD**) F is defined for every logically possible preference profile $(\succsim_1, \succsim_2)$

(**Unanimity, U**) Given a pair of alternatives x, y and a preference profile $\{\succsim_i\}$, if every individual strictly prefers x to y (i.e., $x \succ_i y$), so does the society (i.e., $x \succ y$)

(**Independence of Irrelevant Alternatives, IIA**) Given two preference profiles $\{\succsim_i\}$ and $\{\succsim'_i\}$, if every individual's preference over $\{x, y\}$ remains identical in those profiles, then the social preference over $\{x, y\}$ also remains identical: If $(\succsim_1, \succsim_2) \xrightarrow{F} \succsim$ and $(\succsim'_1, \succsim'_2) \xrightarrow{F} \succsim'$ and if $\forall i (x \succsim_i y \Leftrightarrow x \succsim'_i y) \wedge (y \succsim_i x \Leftrightarrow y \succsim'_i x)$, then $(x \succsim y \Leftrightarrow x \succsim' y) \wedge (y \succsim x \Leftrightarrow y \succsim' x)$

A group of individual V is said to be a **decisive coalition of** F provided that given a pair of alternatives x, y and a profile $\{\succsim_i\}$, if every individual $i \in V$ strictly prefers x to y (i.e., $x \succ_i y$), so does the society (i.e., $x \succ y$). If V consists of a single individual t, i.e., $V=\{t\}$, t is said to be a **dictator.** Note that $\varnothing$ is not decisive since the antecedent is false and that U renders the society T decisive.

Observing that a social welfare function is in our terminology a collective choice rule the range of which is restricted to the set of orderings, Arrow's Paradox states that every Arrovian social welfare function, though merely satisfying a set of innocuous conditions, admits a dictator. In the sequel we denote the set of decisive coalitions of F by $\mathcal{V}_F$.

Mathematical structures of $\mathcal{V}_F$ when F is an Arrovian social welfare function

It is known that in a society T with finitely many individuals, where there are at least three alternatives, if F is an Arrovian social welfare function, then $\mathcal{V}_F$ admits the following properties:

(1) $\varnothing \notin \mathcal{V}_F$ (2) $T \in \mathcal{V}_F$
(3) $U \in \mathcal{V}_F \wedge U \subset W \Rightarrow W \in \mathcal{V}_F$ (4) $U, V \in \mathcal{V}_F \Rightarrow U \cap V \in \mathcal{V}_F$
(5) $V \in \mathcal{V}_F \vee V^c \in \mathcal{V}_F$

A family of subsets $\mathcal{V} \subseteq 2^T$ (2^T denotes the set of all subsets of T) satisfying (1)-(4) is said to be a **filter**, and a filter $\mathcal{V}$ satisfying (5) is said to be an **ultrafilter** (for general discussion on filters and ultrafilters, see for example Sikorski, 1969).

Consider a collection of subsets of T given by $\langle t \rangle = \{V \subseteq T : t \in V\}$, where $t \in T$. It can easily be verified that $\langle t \rangle$ is an ultrafilter. An ultrafilter $\mathcal{V}$ is said to be **fixed at** $t \in T$ if it has the form $\mathcal{V} = \langle t \rangle$ for some $t \in T$.

Note that if T is a finite set, V is an ultrafilter if and only if $V = \langle t \rangle$ is for some $t \in T$: The "only if" implication follows from (5) above which allows any set $V \in \mathcal{V}$ to shrink down to a single point $\{t\} \in \mathcal{V}$ while the validity of the "if" implication has already been mentioned.

Arrow's paradox can be stated in the following alternative form: If T consists of finitely many individuals and if there are at least three alternatives, for any Arrovian social welfare function F, V_F is an ultrafilter.

Observe that V_F is an ultrafilter if and only if $V_F = \langle t \rangle$ is for some $t \in T$ if and only if F admits a dictator $t \in T$, i.e., $\{t\} \in V_F$ (necessary unique by (1) and (4)).

It is interesting to see to what extent Arrow's paradox persists if T is enlarged to a society with infinitely many individuals. Fishburn (1970) showed that with an infinite society T, there is an Arrovian social welfare function that resolves Arrow's paradox, i.e., that admits no dictatorship. Such an Arrovian social welfare function F possesses $\mathcal{V}_F$, an ultrafilter, which is not fixed at any $t \in T$. An ultrafilter $\mathcal{V}$ which is not fixed at any $t \in T$ is said to be **free.** At this point, the following question naturally arises: With an infinite society T, how many of those Arrovian social welfare functions F possess $\mathcal{V}_F$ that are free ?

Hansson (1976) deduced by simple cardinality arguments that if there are only finitely many alternatives, there are as many F with $\mathcal{V}_F$ free as F with $\mathcal{V}_F$ fixed and F with $\mathcal{V}_F$ free combined, suggesting that *the subset consisting of dictatorial Arrovian social welfare functions may be "very small" in the totality of all Arrovian social welfare functions.*

Though our work is inspired by Hansson (1976), we must remark that cardinality comparisons of an infinite total set with its infinite subsets do not quite match with our intuitive, geometric size comparisons. For instance, a plane is intuitively larger than a line contained in it yet they both have the same cardinality. This problem suggests that we need to impose some sort of mathematical structure on the total space, a topology to be exact, which provides a notion of sizes even on infinite subsets.

Ultrafilter property and strict preference property of Arrovian collective choice rules

An Arrovian collective choice rule F is said to satisfy the **ultrafilter property** if $\mathcal{V}_F$ is an ultrafilter. We know from the previous sections that if F is a social welfare function, it satisfies the ultrafilter property. It is interesting to note that if F is complete and merely quasi-transitive, $\mathcal{V}_F$ still is a filter but may not be an ultrafilter and hence may not satisfy the ultrafilter property (Hansson, 1976). An Arrovian collective choice rule F is said to satisfy the **strict preference property** (Torris, 2003, p. 5) if it transforms individual strict preferences into a social strict preference. The last property can be stated formally as follows: Given a pair of alternatives x, y and a profile $\{\succsim_i\}$, if every individual i strictly prefers x to y or y to x (i.e., $x \succ_i y \vee y \succ_i x$), so does the society (i.e., $x \succ y \vee y \succ x$).

Suppose an Arrovian collective choice rule F satisfies the ultrafilter property and given a pair x, y and a profile $\{\succsim_i\}$, every individual i strictly prefers x to y or y to x. Then since $\mathcal{V}_F$ is an ultrafilter, (5) implies that either $\{i : x \succ_i y\}$ or $\{i : y \succ_i x\}$ must be decisive. It follows that $x \succ y$ or $y \succ x$ and consequently, F satisfies the strict preference property.

In the sequel, we denote the set of all Arrovian collective choice rule F satisfying the ultrafilter property by $\mathbb{CR}^{UF}$ and the the set of all Arrovian collective choice rule F satisfyingly the strict preference property by $\mathbb{CR}^{SP}$. We have shown that the following inclusion holds: $\mathbb{CR}^{UF} \subseteq \mathbb{CR}^{SP} \subseteq \mathbb{CR}$, where $\mathbb{CR}$ stands for the totality of all Arrovian collective choice rules. We will show that $\mathbb{CR}^{UF}$ is "very small" in $\mathbb{CR}$ by showing, in light of the above inclusion, that $\mathbb{CR}^{SP}$ is "very small" in $\mathbb{CR}$. One advantage in dealing with $\mathbb{CR}^{SP}$ is that it is mathematically more tractable than $\mathbb{CR}^{UF}$, as we demonstrate in the later sections.

How can we conclude that $\mathbb{CR}^{SP}$ is "very small" in $\mathbb{CR}$?

A topology on a set defines what "open subsets" are in the set, where open subsets are something like open intervals in the real line (for general discussions on topology, see for example Bourbaki, 1966). Once a set is endowed with a topology, we can define a subset to be "very small" if it contains no open subsets (such a subset is called a subset with no interior points). A bit of reflection reveals that a subset with no interior points still doesn't quite match with our intuitive notion of a very small subset. For instance, the set of rational numbers is a subset of the real line and has no interior points but is widely spread and ubiquitous in the real line. To remedy this problem, we strengthen the former condition by insisting that a subset A is "very small" if and only if the smallest closed subset (the complement of an open subset) $\bar{A}$ (called the closure of A) containing A has no interior points. Such a subset is said to be nowhere dense. We can now state precisely one of our goals in this paper, that is, to prove that $\mathbb{CR}^{SP}$ is nowhere dense in $\mathbb{CR}$, which implies that $\mathbb{CR}^{UF}$ is nowhere dense in $\mathbb{CR}$ since the smallest closed subset containing $\mathbb{CR}^{SP}$ certainly contains the smallest closed subset containing $\mathbb{CR}^{UF}$. (In fact, both $\mathbb{CR}^{SP}$ and $\mathbb{CR}^{UF}$ turn out to be closed subsets with respect to the topology we choose).

The topology of pointwise convergence on $\mathbb{CR}$

Let X be a set and $\{0, 1\}$ be a two-point set with the discrete topology (the topology in which every subsets are open). We consider the space $\mathcal{F}$ of all functions $f: X \to \{0, 1\}$. An open subset containing a "point" $f \in \mathcal{F}$, denoted by U_f, is called a neighborhood of f. To specify a topology, we only need to specify the so-called basic open subsets. A neighborhood U_f of $f \in \mathcal{F}$ which is also basic is called a basic neighborhood of f. To deduce that a subset $\mathcal{G} \subseteq \mathcal{F}$ has no interior points, we only need to show that for each point $f \in \mathcal{G}$, no basic neighborhood U_f of f is included in $\mathcal{G}$. It follows by definition that with the topology of pointwise convergence, a basic neighborhood U_f of f takes the form

$$U_f = \{g \in \mathcal{F} : g(x_1) = f(x_1), \ldots, g(x_N) = f(x_N)\}$$

for some finitely many points $x_1, \ldots, x_N$ in X. In other words, U_f consists of all the functions $\mathcal{G}: X \to \{0, 1\}$ which agree with f at finitely many points in the domain X.

Note that since a social preference relation $\succsim$ is a subset of $A \times A$, by considering its characteristic function, it can be viewed as a $\{0, 1\}$-valued function on $A \times A$, i.e., $\succsim \in \{0, 1\}^{A \times A}$ (the space of all functions $f: A \times A \to \{0, 1\}$).

Let Γ be the set of all preference profiles $\{\succsim_i\}$. Then $F \in \mathbb{CR}$ is a function from Γ to $\{0, 1\}^{A \times A}$, which can be viewed as a function $F: \Gamma \times A \times A \to \{0, 1\}$ via the identification $F(\{\succsim_i\}, x, y) \equiv$

$F(\{\succsim_i\})(x, y)$. We now endow the space of all functions $F: \Gamma \times A \times A \to \{0, 1\}$ with the topology of pointwise convergence so that a basic neighborhood $U_F \subseteq \mathbb{CR}$ of F looks like

$$U_F = \left\{ \begin{matrix} G \in \mathbb{CR} : G(\{\succsim_i\}_1, x_1, y_1) = F(\{\succsim_i\}_1, x_1, y_1), \\ \ldots, G(\{\succsim_i\}_N, x_N, y_N) = F(\{\succsim_i\}_N, x_N, y_N) \end{matrix} \right\},$$

where $(\{\succsim_i\}_1, x_1, y_1), \ldots, (\{\succsim_i\}_N, x_N, y_N)$ are some finitely many points in $\Gamma \times A \times A$. With these preparations comleted we proceed to prove our assertion.

$\mathbb{CR}^{SP}$ is nowhere dense in $\mathbb{CR}$ if $|A| = \infty$ and $|T| \geq 2$.

In the sequel, we assume that $\mathbb{CR}$ is endowed with the topology of pointwise convergence. First, we show that $\mathbb{CR}^{SP}$ is a closed subset of $\mathbb{CR}$. To this end, let $F \in \mathbb{CR} \setminus \mathbb{CR}^{SP}$. Then there exist $\{\succsim_i\}_0$ and distinct x_0 and y_0 such that $x_0 \succ_i y_0$ or $y_0 \succ_i x_0$ for all $i \in T$ while $x_0 \not\succ y_0$ and $y_0 \not\succ x_0$, i.e., $F(\{\succsim_i\}_0, x_0, y_0) = 0 \vee F(\{\succsim_i\}_0, y_0, x_0) = 1$ and $F(\{\succsim_i\}_0, y_0, x_0) = 0 \vee F(\{\succsim_i\}_0, x_0, y_0) = 1$. Define

$$U_F = \left\{ \begin{matrix} G \in \mathbb{CR} : G(\{\succsim_i\}_0, x_0, y_0) = F(\{\succsim_i\}_0, x_0, y_0) \\ \wedge\ G(\{\succsim_i\}_0, y_0, x_0) = F(\{\succsim_i\}_0, y_0, x_0) \end{matrix} \right\}.$$

Note that U_F is a basic neighborhood of F in $\mathbb{CR}$ such that $U_F \cap \mathbb{CR}^{SP} = \varnothing$, hence the complement of $\mathbb{CR}^{SP}$ is an open subset, i.e., $\mathbb{CR}^{SP}$ is a closed subset.

Next, we show that $\mathbb{CR}^{SP}$ has no interior points in $\mathbb{CR}$. Let $F \in \mathbb{CR}^{SP}$ and let U_F be a basic neighborhood of F in $\mathbb{CR}$. Then since $|A| = \infty$, we can safely choose $x_0 \neq y_0$ such that both (x_0, y_0) and (y_0, x_0) lie in the complement of the list $\{(x_1, y_1), \ldots, (x_N, y_N)\}$. Now define a deformation $\tilde{F}: \Gamma \times A \times A \to \{0, 1\}$ of F by

$$\tilde{F}(\{\succsim_i\}, x, y) = \begin{cases} 1 \text{ on } \{(x_0, y_0), (y_0, x_0)\} \text{ if } \{\succsim_i\} \text{ satisfies} \\ x_0 \succ_{i_0} y_0 \wedge \forall i \neq i_0 y_0 \succ_i x_0 \\ F(\{\succsim_i\}, x, y) \text{ otherwise,} \end{cases}$$

where $i_0 \in T$. Observe that $T \setminus \{i_0\} \neq \varnothing$ since $|T| \geq 2$. In the sequel, we denote a social preference with respect to $F(\{\succsim_i\})$ by $\overset{F}{\succ}$ and with respect to $\tilde{F}(\{\succsim_i\})$ by by $\overset{\tilde{F}}{\succ}$. We verify the following properties of $\tilde{F}$:

(1) $\tilde{F}$ clearly satisfies UD.

(2) $\tilde{F}$ satisfies U: We must show that if $x \succ_i y$ for all i then $x \overset{\tilde{F}}{\succ} y$. Observe that the only possibly problematic cases occur when $(x, y) = (x_0, y_0)$ or (y_0, x_0). So suppose $x_0 \succ_i y_0$ for all i. Because $|T| \geq 2$, the above $\{\succsim_i\}$ does not satisfy the condition $x_0 \succ_{i_0} y_0 \wedge \forall i \neq i_0\ y_0 \succ_i x_0$. Thus $\tilde{F}(\{\succsim_i\}, x_0, y_0) = F(\{\succsim_i\}, x_0, y_0)$, and since F satisfies U and so $x_0 \overset{F}{\succ} y_0$, it follows that $x_0 \overset{\tilde{F}}{\succ} y_0$. We can also show in a

similar fashion that if $y_0 \succ_i x_0$ for all i then $y_0 \overset{\tilde{F}}{\succ} x_0$ holds. (Note that this is the very point where the condition $|T| \geq 2$ is indispensable).

(3) $\tilde{F}$ satisfies IIA: Suppose that if $\forall i\ (x \succsim_i y \Leftrightarrow x \succsim'_i y) \wedge (y \succsim_i x \Leftrightarrow y \succsim'_i x)$, then $(x \succsim y \Leftrightarrow x \succsim' y) \wedge (y \succsim x \Leftrightarrow y \succsim' x)$. Then we have $\tilde{F}(\{\succsim'_i\}, x, y) = \tilde{F}(\{\succsim''_i\}, x, y)$ and $\tilde{F}(\{\succsim'_i\}, y, x) = \tilde{F}(\{\succsim''_i\}, y, x)$ except possibly for the case $(x, y) = (x_0, y_0)$ and at least one of the preference profiles $\{\succsim'_i\}, \{\succsim''_i\}$ satisfies the condition $x_0 \succ_{i_0} y_0 \wedge \forall i \neq i_0\ y_0 \succ_i x_0$, or the case $(x, y) = (y_0, x_0)$ and at least one of the preference profiles $\{\succsim'_i\}, \{\succsim''_i\}$ satisfies the condition $y_0 \succ_{i_0} x_0 \wedge \forall i \neq i_0 x_0 \succ_i y_0$. So suppose that for all i, $(x_0 \succsim'_i y_0 \Leftrightarrow x_0 \succsim''_i y_0) \wedge (y_0 \succsim'_i x \Leftrightarrow y_0 \succsim''_i x_0)$ and that $x_0 \succ'_{i_0} y_0 \wedge \forall i \neq i_0 y_0 \succ'_i x_0$. Then $x_0 \succ''_{i_0} y_0 \wedge \forall i \neq i_0 y_0 \succ''_i x_0$, and so $\tilde{F}(\{\succsim'_i\}, x_0, y_0) = 1 = \tilde{F}(\{\succsim''_i\}, x_0, y_0)$ and $\tilde{F}(\{\succsim'_i\}, y_0, x_0) = 1 = \tilde{F}(\{\succsim''_i\}, y_0, x_0)$ as desired. The second case can be treated in a similar fashion.

(4) $\tilde{F}$ does not satisfy the strict preference property: Choose any $\{\succsim_i\} \in \Gamma$ which satisfies the conditions $x_0 \succ_{i_0} y_0 \wedge \forall i \neq i_0 y_0 \succ_i x_0$. Then

$$\tilde{F}(\{\succsim_i\}, x_0, y_0) = 1 = \tilde{F}(\{\succsim_i\}, y_0, x_0)$$

while $x_0 \succ_{i_0} y_0 \wedge \forall i \neq i_0 y_0 \succ_i x_0$. Hence $\tilde{F} \notin \mathbb{CR}^{SP}$.

We now deduce that $\tilde{F} \in \mathbb{CR} \setminus \mathbb{CR}^{SP}$ and also that $\tilde{F} \in U_F$. Thus F cannot be an interior point and since $F \in \mathbb{CR}^{SP}$ is arbitrary, $\mathbb{CR}^{SP}$ has no interior points.

So far, we did not impose any sort of collective rationality on $F \in \mathbb{CR}$ and it is not at all clear whether a similar result holds when a total space is restricted to a subspace of $\mathbb{CR}$ consisting of those F satisfying a certain collective rationality such as transitivity, quasi-transitivity, and acyclicity, etc. Our method rests squarely on the fact that the deformation $F \to \tilde{F}$ preserves Arrovian properties, i.e., UD, U, and IIA, and whether or not a similar result holds with a restricted class satisfying a collective rationality solely depends on whether or not the deformation preserves the collective rationality in question. Observe that the transformation will clearly preserve completeness while it may destroy transitivity: Suppose F is transitive and $x_0 \overset{F}{\succ} y_0$, $z \overset{F}{\succ} y_0$, and $x_0 \overset{F}{\succ} z$, where z is distinct from x_0 and y_0. Then it may happen that $x_0 \overset{\tilde{F}}{\sim} y_0$, $z \overset{\tilde{F}}{\succ} y_0$, and $x_0 \overset{\tilde{F}}{\succ} z$, which shows that $\tilde{F}$ is no longer transitive since $z \overset{\tilde{F}}{\succ} y_0$ and $y_0 \overset{\tilde{F}}{\sim} x_0$ does not imply $z \overset{\tilde{F}}{\succsim} x_0$. In contrast, the deformation does preserve acyclicity: Recall that $\succsim$ is said to be acyclic provided that there are no strict preference cycles such as $x_1 \succ x_2 \succ \ldots \succ x_N \succ x_1$ and imposing two alternatives x_0, y_0 to be indifferent could not possibly introduce a strict preference cycle where there were none before. We thus infer that the set of all complete, acyclic, Arrovian collective choice rules satisfying the strict preference property, $\mathbb{CR}^{C, AC, SP}$ for short, is nowhere dense in the set of all complete, acyclic, Arrovian collective choice rules, $\mathbb{CR}^{C,AC}$ for short. The last assertion evidently implies the following:

Theorem 1 *If* $|A|=\infty$ *and* $|T|\geq 2$, *the set of all complete, acyclic, Arrovian collective choice rules satisfying the ultrafilter property,* $\mathbb{CR}^{C,AC,UF}$ *for short, is nowhere dense in the set of all complete, acyclic, Arrovian collective choice rules,* $\mathbb{CR}^{C,AC}$ *for short.*

Properties of social choice functions

In this section we discuss an application of our theorem to social choice problems. A **choice function** C is, by definition, a function that assigns to each nonempty subset $\varnothing \neq S \subseteq A$ a subset $C(S) \subseteq S$, and C is said to be **finitely complete** if $C(S) \neq \varnothing$ for each finite nonempty subset $\varnothing \neq S \subseteq A$.

C is said to be **rationalizable** by a preference relation $\succsim$ if it can be written as $C(S)=\{x \in S : x \succsim y \forall y \in S\}$. On the other hand, C gives rise to a preference relation $\underset{C}{\succsim}$, called the **revealed preference relation**, defined by $x \underset{C}{\succsim} y \Leftrightarrow \exists\, S\, x \in C(S) \wedge y \in S$. We mention the following two consistency conditions on C, which appear frequently in the literature (Blair, Bordes, Kelly, and Suzumura, 1976):

Chernoff condition A contraction consistency in the following sense: $[x \in C(S) \wedge x \in S' \subseteq S] \Rightarrow x \in C(S')$. In words, if x is to be chosen in S where x is contained in a smaller set $S' \subseteq S$, then x is to be chosen in S' as well.

Generalized Condorcet property An expansion consistency in the following sense: $[x \in S \wedge \forall y \in S\, x \in C(\{x, y\})] \Rightarrow x \in C(S)$. In words, if no $y \in S$ can beat $x \in S$ then x is to be chosen in S.

If C can be rationalized by a preference relation $\succsim$, a simple exercise verifies that C satisfies the Chernoff condition and the Generalized Condorcet property. It can also be shown that if C satisfies these two conditions then C can be rationalized by its revealed preference relation $\underset{C}{\succsim}$: First, let $x \in C(S)$ and $y \in S$, and let $S'=\{x, y\} \subseteq S$. Then by the Chernoff condition, $x \in C(S')$ and hence $x \underset{C}{\succsim} y$. Thus $x \in C(S) \Rightarrow \forall y \in S\, x \underset{C}{\succsim} y$. Conversely, suppose $x \in S$ and $\forall y \in S\, x \underset{C}{\succsim} y$, i.e., $\forall y \in S\, \exists S'\, x \in C(S') \wedge y \in S'$. Note that $\{x, y\} \subseteq S'$ and hence by the Chernoff condition, $x \in C(\{x, y\})$. Then by the generalized Condorcet property, $x \in C(S)$.

A **social choice function** $C_{\{\cdot\}}$ is a function that assigns to each preference profile $\{\succsim_i\} \in \Gamma$ a choice function $C_{\{\succsim_i\}}$. Arrow's conditions for collective choice rules can be modified to obtain analogous conditions for social choice functions as follows:

(Universal Domain, UD) $C_{\{\cdot\}}$ is defined for every logically possible preference profile $\{\succsim_i\} \in \Gamma$

(Unanimity, U) Given a pair of alternatives x, y and a preference profile $\{\succsim_i\}$, if every individual strictly prefers x to y (i.e., $x \succ_i y$), then $C_{\{\succsim_i\}}(\{x, y\}) = \{x\}$

(Independence of Irrelevant Alternatives, IIA) Given two preference profiles $\{\succsim_i\}$ and $\{\succsim_i'\}$, if every individual's preference over x, y remains identical in those profiles, then they induce the identical $C_{\{\cdot\}}$: Suppose $(\succsim_1, \succsim_2) \overset{C}{\rightarrow}_{\{\cdot\}} C_{\{\succsim_i\}}$ and $(\succsim_1', \succsim_2') \overset{C}{\rightarrow}_{\{\cdot\}} C_{\{\succsim_i'\}}$. Then
$\forall i\ (x \succsim_i y \Leftrightarrow x \succsim_i' y) \wedge (y \succsim_i x \Leftrightarrow y \succsim_i' x) \Rightarrow C_{\{\succsim_i\}}(\{x, y\}) = C_{\{\succsim_i'\}}(\{x, y\})$

We define a dictator in a similar fashion as in collective choice rules: An individual $t \in T$ is called a **dictator** of $C_{\{\cdot\}}$ provided that given a pair of alternatives x, y and a preference profile $\{\succsim_i\}$, if $x \succ_t y$ then $C_{\{\succsim_i\}}(\{x, y\}) = \{x\}$. Observe that if $C_{\{\succsim_i\}}$ is finitely complete and satisfies the generalized Condorcet property and that if $x \in S$ is the most preferred alternative for a dictator t then $x \in C_{\{\succsim_i\}}(S)$: Suppose $x \in S$ and there is no $y \in S$ such that $y \succ_t x$, which means that there is no $y \in S$ such that $C_{\{\succsim_i\}}(\{x, x\}) = \{y\}$. Then $\forall y \in S\ x \in C_{\{\succsim_i\}}(\{x, x\})$ and hence $x \in C_{\{\succsim_i\}}(S)$. In words, if a dictator thinks an alternative x is best for him in a given agenda S, then society T chooses that alternative x in the agenda S.

It is known that for a rationalizable $C_{\{\succsim_i\}}$, $C_{\{\succsim_i\}}$ is finitely complete if the rationalization $\succsim$ is complete and acyclic, i.e., the underlying collective choice rule $F: \{\succsim_i\} \rightarrow \succsim$ belongs to $\mathbb{CR}^{C, AC}$ (Sen, 1986, p. 1079).

Blair, Bordes, Kelly, and Suzumura (1976, Theorem 2, p. 367) showed that if C satisfies the Chernoff condition and the generalized Condorcet property, then $\underset{C}{\succsim}$ is acyclic and complete.

From what we have discussed so far, we deduce that

(1) Each collective choice rule $F \in \mathbb{CR}^{C, AC}$ defines a finitely complete Arrovian social choice function $C_{\{\cdot\}}$ satisfying the Chernoff condition and the generalized Condorcet property.

(2) Every finitely complete Arrovian social choice function $C_{\{\cdot\}}$ satisfying the Chernoff condition and the generalized Condorcet property arises as an associated social choice function of some collective choice rule $F \in \mathbb{CR}^{C, AC}$.

In the sequel we denote the set of all finitely complete Arrovian social choice function $C_{\{\cdot\}}$ satisfying the Chernoff condition and the generalized Condorcet property by $\mathbb{CF}^{CC, GCP}$. It is a routine matter to verify that the correspondence $\mathbb{CR}^{C, AC} \rightarrow \mathbb{CF}^{CC, GCP}$ given above is, in fact, a bijection. Note that when C satisfies the Chernoff condition, the revealed preference relation $\underset{C}{\succsim}$

admits the following simple description : $x \underset{C}{\succsim} y \Leftrightarrow x \in C(\{x, y\})$, and hence $C_{\{\cdot\}}$ can be viewed as a function $C_{\{\cdot\}} : \Gamma \times A \times A \rightarrow \{0, 1\}$, i.e., $C_{\{\cdot\}}(\{\succsim_i\}, x, y) = 1 \Leftrightarrow x \in C_{\{\succsim_i\}}(\{x, y\})$. We then can endow $\mathbb{CF}^{CC, GCP}$ with the topology of pointwise convergence and reiterate the former arguments for collective choice rules to obtain the following corollary :

Corollary 1 *If $|A| = \infty$ and $|T| \geq 2$, the set of all finitely complete Arrovian social choice functions satisfying the Chernoff condition, the generalized Condorcet property, and the ultrafilter property, $\mathbb{CF}^{CC, GCP, UF}$ for short, is nowhere dense in the set of all finitely complete Arrovian social choice functions satisfying the Chernoff condition and the generalized Condorcet property, $\mathbb{CF}^{CC, GCP}$ for short.*

Social Choice of Market Mechanisms

In this section we focus on a social choice of market mechanisms. As an example, we consider a price adjusting mechanism of an asset market in which, for simplicity, a single asset is assumed to be traded. Thus the price space is $\mathcal{P} = \{x : x > 0\}$ and each trader $i \in T$ is endowed with an ordering $\succsim_i$ on $\mathcal{P}$ the interpretation of which will be given in due course.

We assume that the market quotes an **initial price** p_0 at the outset and then traders seek profits by trading, where their trading strategies rest on their individual prediction of the **terminal price** p_∞. If the terminal price p_∞ coincides with p_0, $p_0 = p_\infty$ is called a **pairwise equilibrium price** (**PWEP for short**). We interpret an individual ordering $\succsim_i$ as an individual terminal price prediction in the following manner : Regardless of an initial price $p_0 \in \{x, y\}$,

$$x \succ_i y \Leftrightarrow x \text{ is more likely to be } p_\infty \text{ than } y$$
$$x \sim_i y \Leftrightarrow x \text{ and } y \text{ are equally likely to be } p_\infty$$

We impose the following conditions on the market :

- The asset price is divisible.
- Traders do not own initial assets and trade only by borrowing and short selling.
- Traders must close their positions at terminal prices p_∞.
- Bankruptcy is not allowed at settlements.
- In the rising phase of the asset price, a borrowing restriction is enforced, and in the falling phase of the asset price, a short-selling restriction is enforced.

The following are examples of trading behavior of different types of traders :

eg. 1 A trader i with $200 \succ_i 100$ believes that $p_\infty = 200$ regardless of whether $p_0 = 100$ or 200. Thus i is willing to buy as long as the price is below 200 and looses incentive to trade when the price reaches 200.

eg. 2 A trader i with $200 \sim_i 100$ is indifferent about $p_\infty = 200$ regardless of whether $p_0 = 100$ or 200.

Thus i is willing to buy when the price is 100 but will not trade as soon as the price goes up by some positive amount $\epsilon > 0$ to $100 + \epsilon$ since bankruptcy results at the settlement in case $p_\infty = 100$.

We define a "social preference relation" $\succsim$ on $\mathcal{P}$ by

$$x \succ y \Leftrightarrow (x \text{ is a PWEP}) \wedge (y \text{ is not a PWEP})$$
$$x \sim y \Leftrightarrow (x \text{ is a PWEP}) \wedge (y \text{ is a PWEP})$$

The following examples ($|T|=3$ is assumed) demonstrate how the market price mechanism works with respect to different individual profiles:

eg. 3 Suppose that $200 \succ_1 100$, $200 \succ_2 100$, $200 \succ_3 100$, and $p_0 = 100$. Then every trader wants to buy at 100 but no trader wants to sell at 100 or higher. Consequently, the market raises the price gradually until it reaches 200 and hence $p_\infty = 200$. Thus 100 is not a pairwise equilibrium price. On the other hand, if $p_0 = 200$ then no trader has incentive to trade and so $p_\infty = 200$ results. Thus 200 is a pairwise equilibrium price while 100 is not, i.e., $200 \succ 100$.

eg. 4 Suppose that $200 \succ_1 100$, $200 \sim_2 100$, $200 \succ_3 100$, and $p_0 = 100$. Then trader 1 and trader 2 want to buy at 100 but no trader wants to sell at 100. Consequently, the market raises the price to say 110. Then trader 1 is willing to buy at 110 while trader 2 cannot afford buying at 110 and trader 3 wants to sell at 110. Since borrowing is limited in the rising phase, the amount of selling at 110 will exceed that of buying at 110. Note that the market cannot raise the price further since the only possible buyer, trader 1, is left with no desire to buy at any price above 110. The market will lower the price to say 105. Then trader 3 is willing to sell at 105 and trader 1 may buy at 105 but since, again, the amount of selling at 105 exceeds that of buying at 105, a similar process continues until $p_\infty = 100$ is reached and hence 100 is a pairwise equilibrium price. If $p_0 = 200$ analogous reasoning reveals that $p_\infty = 200$ and consequently, $200 \sim 100$ follows.

In the sequel, we write x *Pareto* y to mean that for all trader $i \in T$ $x \succsim_i y$ and for at least one $i_0 \in T$ $x \succ_{i_0} y$. From the above examples, we infer that the market price mechanism possesses the following properties:

Property 1. x *Pareto* $y \Rightarrow x \succ y$

Property 2. *not* x *Pareto* $y \Rightarrow y \succsim x$

It is a simple exercise to deduce from the above properties that

(1) x *Pareto* $y \Leftrightarrow x \succ y$, and

(2) *not* x *Pareto* $y \Leftrightarrow y \precsim x$,

i.e. $\precsim$ is the Parato extension rule (Bossert and Suzumura, p. 8, 2009) and hence is complete and quasi-transitive (but not transitive). In particular, $F:\{\precsim_i\} \rightarrow \precsim$ is a member of $\mathbb{CR}^{AC, C}$ and as such generates $C_{\{\cdot\}} \in \mathbb{CF}^{CC, GCP}$ such that for each $\{\precsim_i\}$, $C_{\{\precsim_i\}}$ chooses from a nonempty finite $S \subseteq \boldsymbol{\mathcal{P}}$ the prices $p \in S$ that are pairwise equilibrium prices against any other prices in S.

Concluding Remarks

Observe that a dictator $t \in T$ for $C_{\{\cdot\}} \in \mathbb{CF}^{CC, GCP}$ can, within a given restricted price range S, always render his predicted market prices prevalent no matter how the predictions of the others alter. We remark that since the Pareto extension rule F clearly does not satisfy the strict preference property, the associated social choice function $C_{\{\cdot\}}$ does not admit a dictator. The current discussions indicate that a market mechanism with such a dictator is “rare” and even if it exists, it is not robust against a small perturbation (i.e., a market in which a particular trader can always correctly predict prices exists only rarely and unstably).

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